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The two-body quantum mechanical problem on spheres

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Abstract
The quantum mechanical two-body problem with a central interaction on the sphere $\mathbf{S}^n$ is considered. Using recent results in representation theory, an ordinary differential equation for some energy levels is found. For several interactive potentials these energy levels are calculated in explicit form.

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1. Introduction

The history of mechanics on constant curvature spaces began more than one and a half centuries ago. The analogue of Newton (or Coulomb) force for the hyperbolic space $\mathbf{H}^3$ had already been proposed by the founders of the hyperbolic geometry Lobachevski (between 1835 and 1838) [1] and Bolyai (between 1848 and 1851) [2] as the value $F(\rho)$, which is inverse to the area of the sphere in $\mathbf{H}^3$ of radius ρ with an attractive body in the centre.

The analytical expression for the Newtonian potential in the space $\mathbf{H}^3$ was written in 1870 by Schering [3] (see also his paper [4] of 1873), without any motivation or references to Lobachevski and Bolyai.

In 1873, Lipschitz considered a one-body motion in a central potential on the sphere $\mathbf{S}^2$ [5]. He knew that the central potential V_c satisfies the Laplace equation on $\mathbf{S}^3$. However, for some reason he preferred to consider another central potential $V(\rho) \sim \sin^{-1}(\rho/R)$, where ρ is the distance from the centre and R is the curvature radius. He calculated the general solution of this problem through elliptic functions.

In 1885, Killing found the generalization of all three Kepler laws for the sphere $\mathbf{S}^3$ [6]. He considered the attractive force as an inverse area of a two-dimensional sphere in $\mathbf{S}^3$ as Lobachevski and Bolyai did before. In the following year, these results were also published by Neumann in [7]. Their expansion onto the hyperbolic case was carried out in the Liebmann paper [8] in 1902 and later in 1905 in his book on non-Euclidean geometry [9]. Note that he started from ellipses in $\mathbf{S}^3$ or $\mathbf{H}^3$ and derived a potential in such a way that the first Kepler law would be valid. He also derived the generalization of the oscillator potential for these spaces from the requirement that a particle motion occurs along an ellipse with its centre coinciding with the centre of the potential.

Also in the paper [6], Killing proved the variable separation in the two-centre Kepler problem on the sphere $\mathbf{S}^n$, which implies the integrability of this problem.

The well-known Bertrand theorem [10] states that up to an arbitrary factor there are only two central potentials in Euclidean space that make all bounded trajectories of a one-particle problem closed. In spaces $\mathbf{S}^2, \mathbf{H}^2$ there are also only two potentials V_c and V_o with this property. It was proved by Liebmann in 1903 [11], see also [9].

One can consider the classical mechanics in spaces of constant curvature as a predecessor of special and general relativity. After the rise of these theories, the above papers of Schering, Killing and Liebmann were almost completely forgotten. Note that the description of a particle motion in central potentials in spaces $\mathbf{S}^3$ and $\mathbf{H}^3$ was shortened in the second and the third editions of the Liebmann book [9] with regard to the first edition in favour of special relativity.

Similar models attracted attention later from the point of view of quantum mechanics and the theory of integrable dynamical systems. This led to the rediscovery of the results described above in many papers. Note however that almost forgotten results of Schering, Killing and Liebmann were described in the survey [12].

The quantum mechanical spectral problem on the sphere $\mathbf{S}^3$ for potential V_c (Coulomb problem) was solved by Schrödinger in 1940 by the factorization (ladder) method, invented by himself [13]. Stevenson in 1941 solved the same problem using more traditional analysis of the hypergeometric differential equation [14] (see also Infeld result in 1941 [15]). Infeld and Schild in 1945 solved a similar problem in the space $\mathbf{H}^3$ [16] (see also [17]).

The connection of the Runge–Lenz operator for the quantum Kepler problem in $\mathbf{S}^3$ with the Schrödinger ladder method was discussed by Barut and Wilson in [18]. In [19], Barut, Inomata and Junker solved the Kepler problem in $\mathbf{S}^3$ and $\mathbf{H}^3$ using functional integration. In papers [20, 21], Otchik considered the one-particle quantum two-centre Coulomb problem in $\mathbf{S}^3$ and found a coordinate system admitting the variable separation. The corresponding ordinary differential equations are those of Heun. In [22–26], there was developed an algebraic approach to one-particle problems for potentials V_c and V_o in spaces $\mathbf{S}^n, \mathbf{H}^n$. Transformations between the Coulomb–Kepler and oscillator problems existing in the Euclidean space were generalized for the sphere in [27]. In [28], Bogush, Kurochkin and Otchik considered Coulomb scattering in the space $\mathbf{H}^3$.

The two-body problem with a central interaction in constant curvature spaces $\mathbf{S}^n$ and $\mathbf{H}^n$ considerably differs from its Euclidean analogue. The variable separation for the latter problem is trivial, while for the former one no central potentials are known that admit a variable separation.

The classical two-body problem with a central interaction in constant curvature spaces was considered for the first time in [29]. Its Hamiltonian reduction to the system with two degrees of freedom was carried out by explicit coordinate calculations. For some potentials, there was proved the solvability of the reduced problem for an infinite period of time.

In [30], the self-adjointness of the quantum two-body Hamiltonian in spaces $\mathbf{S}^2$ and $\mathbf{H}^2$ was studied. Also, some infinite energy series for this Hamiltonian with some central potentials on $\mathbf{S}^2$ were found there in explicit form.

Simply connected constant curvature spheres $\mathbf{S}^n$ and hyperbolic spaces $\mathbf{H}^n$ are representatives of the class of two-point homogeneous Riemannian spaces (TPHRS). Such spaces are characterized by the property that any pair of points can be transformed by means of an appropriate isometry to any other pair of points with the same distance between them.

Equivalently, these spaces are characterized by the property that the natural action of the isometry group on the unit sphere bundle over them is transitive. The classification of TPHRS can be found in [31].

For a smooth manifold M endowed with a left action of a Lie group G, denote by $\mathrm{Diff}(M) \equiv \mathrm{Diff}_G(M)$ the algebra of G-invariant differential operators on M with smooth coefficients. For a Riemannian manifold M let M_S be the unit sphere bundle over M. Let Q be an arbitrary TPHRS, endowed with the action of the identity component of the isometry group for Q.

In [32], there was found a polynomial expression for the quantum two-body Hamiltonian H on Q through a radial differential operator and generators of the algebra $\mathrm{Diff}(Q_S)$. Coefficients of this polynomial depend only on the distance between particles.

Algebras $\mathrm{Diff}(Q_S)$ are noncommutative. A full set of their generators and corresponding relations[1] was found in [33].

Let $\mathfrak{A}$ be a set of $\mathrm{Diff}(Q_S)$ generators presented in the expression for the Hamiltonian H. Every common eigenfunction of operators from $\mathfrak{A}$ generates a separate spectral ordinary differential equation for the two-body quantum mechanical problem on TPHRS. The search for such a common eigenfunction is not an easy problem. In low dimensions for $Q = \mathbf{S}^2$, $Q = \mathbf{S}^3$, this problem was solved in [30] and [34] using an explicit description of $SO(3)$ and $SO(4)$ irreducible representations. The present paper deals with this problem for the general spherical case $Q = \mathbf{S}^n$. Progress is made using the results in representations theory of the algebras $\mathfrak{so}(n, \mathbb{C})$ in [35] and [36].

The paper is organized as follows. Sections 2–4 are of a preparatory character. Sections 2 and 3 contain basic facts on invariant differential operators on homogeneous spaces and regular representations of compact Lie groups, respectively. In section 4, there is a description of the quantum two-body Hamiltonian on the sphere $\mathbf{S}^n$ through a radial differential operator and generators D_i, $i = 0, 1, 2, 3$, of the algebra $\mathrm{Diff}\big(\mathbf{S}^n_S\big)$.

Sections 5 and 6 form the main part of the paper. In section 5, we calculate actions of operators $D_i, i = 0, 1, 2, 3$, in a corresponding functional space and find all common eigenvectors ψ_D for operators D_0^2, D_1, D_2 and optionally D_3. Using these eigenvectors, we derive in section 6 a separate ordinary differential equation of the second order for a radial part of a two-body eigenfunctions. For Coulomb and oscillator potentials this differential equation is Fuchsian and we consider its reducibility to the hypergeometric one using the rational change of an independent variable. This reduction is possible for some eigenvectors ψ_D that lead to an explicit form of some infinite energy level series for the two-body problem with Coulomb and oscillator potentials.

Necessary information concerning complex orthogonal Lie algebras, self-adjoint Schrödinger operators on Riemannian spaces and Fuchsian differential equations is collected in appendices A–C.

2. Invariant differential operators on homogeneous spaces

Here we shall briefly describe the construction of invariant differential operators on homogeneous spaces [37].

Let G be a Lie group of dimension N and K be its subgroup of dimension $N - \ell$. Denote the corresponding Lie algebras as $\mathfrak{g}$ and $\mathfrak{k}$. Suppose that the algebra $\mathfrak{g}$ admits the reductive expansion

$$\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}, \tag{1}$$

[1] One relation for the quaternion projective space and its hyperbolic analogue was calculated only in leading terms.

for a subspace $\mathfrak{p} \subset \mathfrak{g}$, i.e. $\mathrm{Ad}_K \mathfrak{p} \subset \mathfrak{p}$. For a compact Lie group G such subspace $\mathfrak{p}$ can always be constructed using the invariant integration on G. Let $(e_j)_{j=1}^N$ be a base in $\mathfrak{g}$ such that $(e_j)_{j=1}^{\ell}$ is a base in $\mathfrak{p}$.

Let $S(\mathfrak{p})$ be a symmetric algebra for the linear space $\mathfrak{p}$. The Ad_K-action on $\mathfrak{p}$ is naturally extended to the Ad_K-action on $S(\mathfrak{p})$. The main result of the general theory [37] is that G-invariant differential operators on G/K are in one-to-one correspondence with the set $S(\mathfrak{p})^K$ of all Ad_K-invariant elements in $S(\mathfrak{p})$.

Let $\imath : \mathfrak{p} \to S(\mathfrak{p})$ be an inclusion, $U(\mathfrak{g})$ be the universal enveloping algebra for $\mathfrak{g}$ and $\lambda : S(\mathfrak{p}) \to U(\mathfrak{g})$ be a linear symmetrization map, defined on monomials by the formula

$$\lambda\left(e_{i_1}^* \cdots e_{i_k}^*\right) = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} e_{i_{\sigma(1)}} \cdots e_{i_{\sigma(k)}}, \qquad 1 \leqslant i_j \leqslant \ell, \quad 1 \leqslant j \leqslant k,$$

where $e_{i_j}^* := \imath(e_{i_j})$ and $\mathfrak{S}_k$ is the full permutations group of k elements. Evidently,

$$\lambda : S(\mathfrak{p})^K \to U(\mathfrak{g})^K,$$

where $U(\mathfrak{g})^K$ is the set of all Ad_K-invariant elements in $U(\mathfrak{g})^K$.

Let $P(e_1, \ldots, e_N)$ be a polynomial depending on noncommutative elements. Denote by $\tilde{e}_i$ the left invariant vector field on G, corresponding to the element $e_i \in \mathfrak{g} \cong T_e G$:

$$\tilde{e}_i|_g = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g \exp(te_i), \qquad g \in G.$$

Then, $D_P := P(\tilde{e}_1, \ldots, \tilde{e}_N)$ is a left invariant differential operator on G.

Functions on the homogeneous space G/K are in one-to-one correspondence with functions on the group G that are invariant w.r.t. right K-shifts. For $P(e_1, \ldots, e_N) \in U(\mathfrak{g})^K$ the differential operator D_P, acting on such functions, can be considered as a G-invariant differential operator on the space G/K and every such operator can be uniquely represented in the form

$$(\lambda(P_0))(\tilde{e}_1, \ldots, \tilde{e}_\ell)),$$

for some $P_0 \in S(\mathfrak{p})^K$.

3. Regular representations of compact Lie groups

Let G be a compact connected Lie group and μ be a bi-invariant positive measure on G, unique up to an arbitrary factor [38]. Let $\mathcal{L}^2(G, \mu)$ be a Hilbert space of measurable complex-valued functions on G, square integrable w.r.t. the measure μ. Define two unitary left representations of G in the space $\mathcal{L}^2(G, \mu)$. The *left regular representation* T^l acts by the left shifts

$$\left(T_q^l f\right)(g) = f(q^{-1}g), \qquad q, g \in G, \qquad f \in \mathcal{L}^2(G, \mu)$$

and the *right regular representation* T^r acts by the right shifts

$$\left(T_q^r f\right)(g) = f(gq), \qquad q, g \in G, \qquad f \in \mathcal{L}^2(G, \mu).$$

Evidently, these representations are equivalent with the intertwining operator $f(g) \to f(g^{-1})$. It is well known that these representations expand into direct sums of finite-dimensional unitary irreducible representations (irreps). Each of these irreps is contained in T^l or T^r with a multiplicity equal to its dimension and every linear irreducible representation of G is equivalent to an irreps from this sum [39, 40].

Let T_ℓ be a full system of unitary irreps for G in spaces U_ℓ, $\ell = 1, 2, \ldots$. Choose in every U_ℓ an orthonormal base $(e_{\ell,k})_{k=1}^{d_\ell}$, $d_\ell := \dim_{\mathbb{C}} U_\ell$. Define matrix elements $t_{\ell,k}^i$ of operators

T_q^r by the equation $T_q^r e_{\ell,k} =: t^i_{\ell,k}(q)e_{\ell,i}$ or equivalently by $t^i_{\ell,k}(q) := \langle e_{\ell,i}, T_q^r e_{\ell,k}\rangle_{U_\ell}$, $q \in G$. Since

$$t^i_{\ell,k}(gq)e_{\ell,i} = T_g^r T_q^r e_{\ell,k} = t^j_{\ell,i}(g)t^i_{\ell,k}(q)e_{\ell,j}, \qquad g, q \in G$$

one has

$$t^i_{\ell,k}(gq) = t^i_{\ell,j}(g)t^j_{\ell,k}(q). \tag{2}$$

Therefore, the subspace $\mathcal{R}_{\ell,i} \subset \mathcal{L}^2(G, \mu)$, spanned by functions $\left(t^i_{\ell,j}(g)\right)_{j=1}^{d_\ell}$, is invariant under operators T_q^r and the representation $T^r|_{\mathcal{R}_{\ell,i}}$ is equivalent to T_ℓ. On the other hand, formula (2) implies that the subspace $\mathcal{L}_{\ell,j} \subset \mathcal{L}^2(G, \mu)$, spanned by functions $\left(t^i_{\ell,j}(g)\right)_{i=1}^{d_\ell}$, is invariant under operators T_q^l and the representation $T^l|_{\mathcal{L}_{\ell,j}}$ is again equivalent to T_ℓ. The functions $\left(t^i_{\ell,j}(g)\right)_{i,j=1}^{d_\ell}$, $\ell = 1, 2, \ldots$, form an orthogonal base in the space $\mathcal{L}^2(G, \mu)$ [38–40] and

$$\|t^i_{\ell,j}\|^2_{\mathcal{L}^2(G,\mu)} = \frac{\mu(G)}{d_\ell}.$$

Thus, the space

$$\mathcal{T}_\ell := \bigoplus_{i=1}^{d_\ell} \mathcal{R}_{\ell,i} = \bigoplus_{j=1}^{d_\ell} \mathcal{L}_{\ell,j}$$

is invariant under representations T^r and T^l. The representation T^r intermixes spaces $\mathcal{L}_{\ell,j}$ of representations T^l and vice versa the representation T^l intermixes spaces $\mathcal{R}_{\ell,i}$ of representations T^r. The space $\mathcal{L}^2(G, \mu)$ of representations T^r and T^l expands into irreps as follows:

$$\mathcal{L}^2(G, \mu) = \bigoplus_\ell \mathcal{T}_\ell = \bigoplus_\ell \bigoplus_{i=1}^{d_\ell} \mathcal{R}_{\ell,i} = \bigoplus_\ell \bigoplus_{j=1}^{d_\ell} \mathcal{L}_{\ell,j}.$$

For a Lie subgroup K of the group G, the subspace $\mathcal{L}^2(G, K, \mu) \subset \mathcal{L}^2(G, \mu)$, consisting of functions invariant w.r.t. all right K-shifts on G, is invariant w.r.t. left G-shifts. Therefore, there are only two possibilities:

$$\mathcal{L}_{\ell,j} \cap \mathcal{L}^2(G, K, \mu) = \mathcal{L}_{\ell,j} \qquad \text{and} \qquad \mathcal{L}_{\ell,j} \cap \mathcal{L}^2(G, K, \mu) = 0.$$

The consideration above implies the following proposition.

Proposition 1. *Let*

$$\widetilde{\mathcal{T}}_\ell := \mathcal{T}_\ell \cap \mathcal{L}^2(G, K, \mu), \qquad \widetilde{\mathcal{R}}_{\ell,i} := \mathcal{R}_{\ell,i} \cap \mathcal{L}^2(G, K, \mu), \qquad \tilde{d}_\ell := \dim_{\mathbb{C}} \widetilde{\mathcal{R}}_{\ell,i}.$$

Evidently, the value $\tilde{d}_\ell$ does not depend on $i = 1, \ldots, d_\ell$. The representation $T^l|_{\widetilde{\mathcal{T}}_\ell}$ is expanded into the direct sum of equivalent irreps in spaces $\mathcal{L}^K_{\ell,k}$, $k = 1, \ldots, \tilde{d}_\ell$, which are among of $\mathcal{L}_{\ell,j}$. On the other hand,

$$\widetilde{\mathcal{T}}_\ell = \bigoplus_{i=1}^{d_\ell} \widetilde{\mathcal{R}}_{\ell,i},$$

where the spaces $\widetilde{\mathcal{R}}_{\ell,i}$, $i = 1, \ldots, d_\ell$, are isomorphic to each other.

4. Two-body Hamiltonian on the sphere $\mathbf{S}^n$

Let $\mathbf{S}^n$, $n \geqslant 2$, be the n-dimensional sphere, endowed with the standard metric g of a constant sectional curvature R^{-2}, $R > 0$ and

$$\triangle = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^i} \left(\sqrt{\gamma} g^{ij} \frac{\partial}{\partial x^j} \right)$$

be the corresponding Laplace–Beltrami operator, expressed through local coordinates, where $\gamma := \det \| g_{ij} \|$. We start from the description of the two-body quantum Hamiltonian on $\mathbf{S}^n$ found in [32, 41].

The configurations space for the two-body system on $\mathbf{S}^n$ is

$$\mathbf{S}^n \times \mathbf{S}^n. \tag{3}$$

The Hamiltonian for this system is

$$H_V = H_0 + V \equiv -\frac{1}{2m_1} \triangle_1 - \frac{1}{2m_2} \triangle_2 + V(\rho), \tag{4}$$

where $\triangle_i$, $i = 1, 2$, is the Laplace–Beltrami operator on the ith factor of (3) and ρ be the distance between particles. It should be defined on some subspace $\mathrm{Dom}(H)$ dense in $\mathcal{L}^2 (\mathbf{S}^n \times \mathbf{S}^n, \chi \times \chi)$ to be a self-adjoint operator, where χ is the measure on $\mathbf{S}^n$ induced by the metric. In local coordinates, χ has the form $\chi = \sqrt{\gamma}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. Note that the free Hamiltonian H_0 is the Laplace–Beltrami operator for the metric

$$g_2 := m_1 \tilde{\pi}_1^* g + m_2 \tilde{\pi}_2^* g \tag{5}$$

on (3), multiplied by $-1/2$, where $\tilde{\pi}_i^* g$ is the pullback of the metric g with respect to the projection on the ith factor.

Let $G \cong SO(n+1)$ be the identity component of the isometry group for the sphere $\mathbf{S}^n$. One can consider $SO(n+1)$ in the standard way as

$$SO(n+1) = (A \in GL(n+1, \mathbb{R}) | AA^T = E, \det A = 1),$$

where E is the matrix unit. The configuration space (3) is endowed with the diagonal G-action and the differential operator (4) is G-invariant.

Let $K \cong SO(n-1)$ be a subgroup in $SO(n+1)$ with elements of the form

$$\begin{pmatrix} E_2 & 0 \\ 0 & A \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A \in SO(n-1).$$

Up to a manifold of dimension n, consisting of antipodal points, the configuration space (3) can be represented as the direct product

$$I \times (G/K), \tag{6}$$

where $I = (0, \pi R)$ and the factor space G/K is G-homogeneous w.r.t. left shifts [32]. The space G/K is isomorphic to the unit sphere bundle over $\mathbf{S}^n$ [33].

The Lie algebra $\mathfrak{g} \cong \mathfrak{so}(n+1)$, corresponding to the group G, consists of skew-symmetric matrices. Let E_{kj} be the matrix of the size $(n+1) \times (n+1)$ with the unique nonzero element equals 1, locating at the intersection of the kth row and the jth column. Choose the base for the algebra $\mathfrak{g}$ as

$$\Psi_{kj} = E_{kj} - E_{jk}, \qquad 1 \leqslant k < j \leqslant n+1.$$

The algebra $\mathfrak{g}$ admits the reductive expansion (1), where the subspace $\mathfrak{p}$ is spanned by elements

$$\Psi_{1k}, \quad 2 \leqslant k \leqslant n+1, \qquad \Psi_{2k}, \quad 3 \leqslant k \leqslant n+1.$$

In the general case $n \geqslant 4$ generators of the commutative algebra $S(\mathfrak{p})^K$ can be chosen [33] as

$$-\Psi_{12}^*, \qquad \sum_{k=3}^{n+1}(\Psi_{1k}^*)^2, \qquad \sum_{k=3}^{n+1}(\Psi_{2k}^*)^2, \qquad -\sum_{k=3}^{n+1}\Psi_{1k}^*\Psi_{2k}^*.$$

In the case $n = 3$, there is the additional generator

$$\Box^* = \Psi_{13}^*\Psi_{24}^* - \Psi_{14}^*\Psi_{23}^*.$$

In the case $n = 2$, the group K is trivial and generators of $S(\mathfrak{p})^K = S(\mathfrak{p})$ are simply

$$\Psi_{12}^*, \qquad \Psi_{13}^*, \qquad \Psi_{23}^*.$$

In all cases, we shall consider elements

$$D_0 = -\Psi_{12}, \qquad D_1 = \sum_{k=3}^{n+1}\Psi_{1k}^2, \qquad D_2 = \sum_{k=3}^{n+1}\Psi_{2k}^2, \qquad D_3 = -\frac{1}{2}\sum_{k=3}^{n+1}\{\Psi_{1k}, \Psi_{2k}\} \tag{7}$$

from $U(\mathfrak{g})$ as invariant differential operators on the space G/K, where $\{\cdot, \cdot\}$ means an anticommutator. The commutative relations for differential operators (7) are (see [33])

$$\begin{aligned}
&[D_0, D_1] = -2D_3, \qquad [D_0, D_2] = 2D_3, \qquad [D_0, D_3] = D_1 - D_2,\\
&[D_1, D_2] = -2\{D_0, D_3\}, \qquad [D_1, D_3] = -\{D_0, D_1\} + \frac{(n-1)(n-3)}{2}D_0,\\
&[D_2, D_3] = \{D_0, D_2\} - \frac{(n-1)(n-3)}{2}D_0.
\end{aligned} \tag{8}$$

For $n = 3$, the additional operator

$$\Box := \tfrac{1}{2}(\{\Psi_{13}, \Psi_{24}\} - \{\Psi_{14}, \Psi_{23}\})$$

lies in the centre of the algebra $\mathrm{Diff}_G(G/K)$.

Define a new coordinate r on the interval I by the equation

$$r = \tan\left(\frac{\rho}{2R}\right), \qquad r \in \mathbb{R}_+ := (0, \infty).$$

Results from [32, 33] imply the following theorem.

Theorem 1. *The quantum two-body Hamiltonian on the sphere* $\mathbf{S}^n$ *can be considered as the differential operator*

$$\begin{aligned}
H = &-\frac{(1+r^2)^n}{8mR^2r^{n-1}}\frac{\partial}{\partial r}\circ\left(\frac{r^{n-1}}{(1+r^2)^{n-2}}\frac{\partial}{\partial r}\right) - \frac{m_1\alpha^2 + m_2\beta^2}{2m_1m_2R^2}D_0^2\\
&+ \frac{(m_1\alpha - m_2\beta)(1+r^2)^n}{4m_1m_2R^2r^{n-1}}\left\{\frac{\partial}{\partial r}, \frac{r^{n-1}D_0}{(1+r^2)^{n-1}}\right\}\\
&- \frac{1}{2}(CD_1 + AD_2 + 2BD_3) + V(r),
\end{aligned} \tag{9}$$

on the space $\mathbb{R}_+ \times G$*, where*

$$m := \frac{m_1m_2}{m_1 + m_2}, \tag{10}$$

a parameter $\alpha \in (0, 1)$ *is arbitrary,* $\beta := 1 - \alpha$ *and*

$$\begin{aligned}
A &= \frac{(1+r^2)^2}{4m_1m_2R^2r^2}(m_1\cos^2(2\alpha\arctan r) + m_2\cos^2(2\beta\arctan r)),\\
B &= \frac{(1+r^2)^2}{8m_1m_2R^2r^2}(m_1\sin(4\alpha\arctan r) - m_2\sin(4\beta\arctan r)),\\
C &= \frac{(1+r^2)^2}{4m_1m_2R^2r^2}(m_1\sin^2(2\alpha\arctan r) + m_2\sin^2(2\beta\arctan r)).
\end{aligned}$$

The domain for operator (9) is dense in the space $\mathcal{L}^2(\mathbb{R}_+ \times G, K, \eta)$*, consisting of all complex-valued square integrable K-invariant functions on* $\mathbb{R}_+ \times G$*, with respect to right K-shifts and the measure*

$$\mathrm{d}\eta = \frac{r^{n-1}\,\mathrm{d}r}{(1+r^2)^n} \otimes \mathrm{d}\mu \equiv \mathrm{d}\nu \otimes \mathrm{d}\mu,$$

where μ *is a biinvariant measure on G, unique up to a constant factor.*

In the following, we choose the parameter α in such a way that $m_1\alpha - m_2\beta = 0$, i.e.,

$$\alpha = \frac{m_2}{m_1 + m_2}, \qquad \beta = \frac{m_1}{m_1 + m_2}.$$

For such choice operator (9) becomes

$$H = -\frac{(1+r^2)^n}{8mR^2r^{n-1}}\frac{\partial}{\partial r} \circ \left(\frac{r^{n-1}}{(1+r^2)^{n-2}}\frac{\partial}{\partial r}\right) - \frac{1}{2(m_1+m_2)R^2}D_0^2 \\ - \frac{1}{2}(CD_1 + AD_2 + 2BD_3) + V(r). \tag{11}$$

It is obvious that

$$\mathcal{L}^2(\mathbb{R}_+ \times G, K, \eta) = \mathcal{L}^2(\mathbb{R}_+, \nu) \otimes \mathcal{L}^2(G, K, \mu). \tag{12}$$

Operators D_0^2, D_1, D_2, D_3 act on the second factor in (12). This action will be studied in the following section.

Note that $B \equiv 0$ for $m_1 = m_2$. Let $\psi_D \in \mathcal{L}^2(G, K, \mu)$ be a common eigenfunctions for operators D_0^2, D_1, D_2 if $m_1 = m_2$ and also for D_3 if $m_1 \neq m_2$. Then the following stationary Schrödinger equation

$$H(f(r)\psi_D) = Ef(r)\psi_D \tag{13}$$

is equivalent to a spectral problem for an ordinary differential equation for a function $f(r)$ and an energy level E (in other words to a one-dimensional stationary Schrödinger equation).[2]

Proposition 2. *Let* ψ_D *be a common eigenfunction for operators* D_0^2, D_1, D_2, D_3 *with eigenvalues* $\delta_0, \delta_1, \delta_2$ *and* δ_3*, respectively. Then,*

1. $\delta_1 = \delta_2$ *and* $\delta_3 = 0$*;*
2. $D_0\psi_D$ *is an eigenfunction for operators* D_0^2, D_1, D_2, D_3 *with the same eigenvalues* $\delta_0, \delta_1, \delta_2$ *and* δ_3*, respectively;*
3. *if* $D_0\psi_D \not\sim \psi_D$*, then* $D_0\psi_D \pm \sqrt{\delta_0}\psi_D$ *are eigenfunctions for operators* D_0, D_1, D_2, D_3*;*
4. *if* $D_0\psi_D \sim \psi_D$*, then either* $D_0\psi_D = 0$ *or* $\delta_1 = \delta_2 = (n-1)(n-3)/4$*.*

Proof. Relations $[D_0, D_3] = D_1 - D_2$ and $[D_1, D_2] = -2\{D_0, D_3\}$ imply

$$[D_0, D_3]\psi_D = \delta_3 D_0\psi_D - D_3D_0\psi_D = (D_1 - D_2)\psi_D = (\delta_1 - \delta_2)\psi_D, \\ \delta_3 D_0\psi_D + D_3D_0\psi_D = \{D_0, D_3\}\psi_D = -\tfrac{1}{2}[D_1, D_2]\psi_D = 0. \tag{14}$$

The last two equations lead to

$$2\delta_3 D_0\psi_D = (\delta_1 - \delta_2)\psi_D. \tag{15}$$

If $\delta_3 \neq 0$, then the last equation implies $D_0\psi_D \sim \psi_D$ and the relation $[D_0, D_1] = -2D_3$ gives $\delta_3\psi_D = D_3\psi_D = -\frac{1}{2}[D_0, D_1]\psi_D = 0$. Thus, $\delta_3 = 0$ and equation (15) implies $\delta_1 = \delta_2$ that proves the first claim of the proposition.

[2] Note that such eigenfunctions are very special elements of the space $\mathcal{L}^2(G, K, \mu)$ and they do not span it.

Now from equation (14) one gets $D_3D_0\psi_D = 0$ and the first two relations (8) imply $D_1D_0\psi_D = D_2D_0\psi_D = \delta_1 D_0\psi_D$. The relation $D_0^2D_0\psi_D = \delta_0 D_0\psi_D$ is evident, which completes the proof of the second claim.

The relation $D_0^2\psi_D = \delta_0\psi_D$ is equivalent to $(D_0 + \sqrt{\delta_0}\,\mathrm{id})(D_0 - \sqrt{\delta_0}\,\mathrm{id})\psi_D = 0$. Now if $D_0\psi_D \neq \sqrt{\delta_0}\psi_D$, then $\psi_D^- := (D_0 - \sqrt{\delta_0}\,\mathrm{id})\psi_D$ is an eigenfunction for the operator D_0. The function ψ_D^- is also an eigenfunction for operators D_1, D_2, D_3 due to the second claim. The consideration for the function $\psi_D^+ := (D_0 + \sqrt{\delta_0}\,\mathrm{id})\psi_D$ is completely similar. Thus, the third claim is proved.

Assume now $D_0\psi_D = \delta_0'\psi_D$. Then, the last relation from (8) gives

$$2\delta_0'\delta_2\psi_D = \tfrac{1}{2}(n-1)(n-3)\delta_0'\psi_D.$$

It means either $\delta_0' = 0$ or $\delta_1 = \delta_2 = (n-1)(n-3)/4$, which proves the last claim. □

5. Action of operators D_0, D_1, D_2, D_3 in the space $\mathcal{L}^2(G, K, \mu)$

Here, we use the notation of section 3 for $G = SO(n+1)$ and $K = SO(n-1)$. Below we mean by the complexification $\mathfrak{g}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ the following set

$$\mathfrak{so}(n+1,\mathbb{C}) = (A \in \mathfrak{gl}(n+1,\mathbb{C}) \mid A + A^T = E). \tag{16}$$

Operators D_i are polynomial w.r.t. infinitesimal generators of right G-shifts. Therefore, they conserve the spaces $\widetilde{\mathcal{T}}_\ell$ and generally intermix its direct summands $\mathcal{L}^K_{\ell,k}$, $k = 1, \ldots, \tilde{d}_\ell$, with constant ℓ and different k. On the other hand, they act in spaces $\widetilde{R}_{\ell,i}$ and their action is the same for constant ℓ and different $i = 1, \ldots, d_\ell$.

From now we shall treat complex spaces $R_{\ell,i}$ as simple left modules over $\mathfrak{g}^{\mathbb{C}}$. Their subspaces $\widetilde{R}_{\ell,i}$ consist of elements annulled by the subalgebra $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{so}(n-1,\mathbb{C}) \subset \mathfrak{g}^{\mathbb{C}}$.

The classification of such modules based on the notion of a dominant weight is well known [46, 47] (see also appendices A and B for a brief description). In order to apply this theory one should use the form of $\mathfrak{so}(n+1,\mathbb{C})$, described in appendix A and different from (16). Besides, since $\mathfrak{B}_k := \mathfrak{so}(2k+1,\mathbb{C})$ and $\mathfrak{D}_k := \mathfrak{so}(2k,\mathbb{C})$ are different series of simple complex Lie algebras, we shall consider cases of odd and even n separately.

5.1. The case $n = 2k$

In this section, we shall use notation from appendix A.1. In particular, by $\mathfrak{B}_k$ we mean the set (A.1). First of all, we shall construct the isomorphism $\mathfrak{g}^{\mathbb{C}} \cong \mathfrak{B}_k$ in explicit form.

Let

$$J_{2k+1} = \begin{pmatrix} \frac{1}{\sqrt{2}}E_k & 0 & \frac{1}{\sqrt{2}}S_k \\ 0 & 1 & 0 \\ \frac{\mathbf{i}}{\sqrt{2}}S_k & 0 & \frac{-\mathbf{i}}{\sqrt{2}}E_k \end{pmatrix} \in GL(2k+1,\mathbb{C}),$$

where $\mathbf{i}$ is the complex unit. It is easily verified that

$$J_{2k+1}S_{2k+1}J_{2k+1}^T = E_{2k+1}.$$

Therefore, the equation $A^TS_{2k+1} + S_{2k+1}A = 0$ for $A \in \mathfrak{gl}(2k+1,\mathbb{C})$ is equivalent to the equation $B^T + B = 0$, where $B := \left(J_{2k+1}^T\right)^{-1}AJ_{2k+1}^T$. Thus, the map

$$B \to J_{2k+1}^T B\left(J_{2k+1}^T\right)^{-1} \tag{17}$$

is the isomorphism between $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{B}_k$.

Let

$$C = \begin{pmatrix} 0 & \alpha & A_- & a & A_+ \\ -\alpha & 0 & B_- & b & B_+ \\ -A_-^T & -B_-^T & & & \\ -a & -b & & C' & \\ -A_+^T & -B_+^T & & & \end{pmatrix} \in \mathfrak{g},$$

where

$A_- = (a_{-(k-1)}, \dots, a_{-1})$, $\quad A_+ = (a_1, \dots, a_{k-1})$, $\quad B_- = (b_{-(k-1)}, \dots, b_{-1})$,
$B_+ = (b_1, \dots, b_{k-1})$, $\quad a_i, b_i, a, b \in \mathbb{R}$, $\quad C' \in \mathfrak{so}(2k-1)$.

Move the second row and the second column of the matrix C to the last positions. This gives the matrix

$$\widetilde{C} = \begin{pmatrix} 0 & A & a & A_+ & \alpha \\ -A_-^T & & & & -B_-^T \\ -a & & \widetilde{C}' & & -b \\ -A_+^T & & & & -B_+^T \\ -\alpha & B_- & b & B_+ & 0 \end{pmatrix} \in \mathfrak{so}(2k+1), \qquad \widetilde{C}' \in \mathfrak{so}(2k-1).$$

The transformation (17) now gives for $\widehat{C} := J_{2k+1}^T \widetilde{C} \left(J_{2k+1}^T\right)^{-1}$ the expression

$$\widehat{C} = \frac{1}{2} \begin{pmatrix} -2\mathbf{i}\alpha & Z_- - \mathbf{i}Z_+S_{k-1} & \sqrt{2}z & Z_-S_{k-1} + \mathbf{i}Z_+ & 0 \\ -\overline{Z}_-^T - \mathbf{i}S_{k-1}\overline{Z}_+^T & & & & -Z_-^T - \mathbf{i}S_{k-1}Z_+^T \\ -\sqrt{2}\bar{z} & & \widehat{C}' & & -\sqrt{2}z \\ -S_{k-1}\overline{Z}_-^T + \mathbf{i}\overline{Z}_+^T & & & & -S_{k-1}Z_-^T + \mathbf{i}Z_+^T \\ 0 & \overline{Z}_- - \mathbf{i}\overline{Z}_+S_{k-1} & \sqrt{2}\bar{z} & \overline{Z}_-S_{k-1} + \mathbf{i}\overline{Z}_+ & 2\mathbf{i}\alpha \end{pmatrix},$$

where $Z_- := A_- + \mathbf{i}B_-$, $Z_+ := A_+ + \mathbf{i}B_+$, $z := a + \mathbf{i}b$, $\widehat{C}' \in \mathfrak{B}_{k-1}$. Let us identify Lie algebras $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{B}_k$ through the map $C \to \widehat{C}$. Due to the definition of Ψ_{ij} in section 4, one gets the following formulae:

$$\Psi_{12} = \mathbf{i}F_{kk}, \qquad \Psi_{1,k+2} = \frac{1}{\sqrt{2}}(F_{k0} - F_{0k}), \qquad \Psi_{2,k+2} = -\frac{\mathbf{i}}{\sqrt{2}}(F_{k0} + F_{0k}),$$

$$\Psi_{1i} = \frac{1}{2}(F_{kj} + F_{k,-j} + F_{-kj} + F_{-k,-j}), \qquad j = i-k-2, \qquad 3 \leqslant i \leqslant k+1,$$

$$\Psi_{1i} = \frac{\mathbf{i}}{2}(F_{kj} - F_{k,-j} + F_{-kj} - F_{-k,-j}), \qquad j = i-k-2, \qquad k+3 \leqslant i \leqslant 2k+1,$$

$$\Psi_{2i} = \frac{\mathbf{i}}{2}(F_{-kj} + F_{-k,-j} - F_{kj} - F_{k,-j}), \qquad j = i-k-2, \qquad 3 \leqslant i \leqslant k+1,$$

$$\Psi_{2i} = \frac{1}{2}(F_{kj} - F_{k,-j} + F_{-k,-j} - F_{-kj}), \qquad j = i-k-2, \qquad k+3 \leqslant i \leqslant 2k+1,$$

which imply

$$D_1 = \frac{1}{2}(F_{k0} - F_{0k})^2 + \frac{1}{2}\sum_{j=1}^{k-1}\{F_{-kj} + F_{kj}, F_{k,-j} + F_{-k,-j}\},$$

$$D_2 = -\frac{1}{2}(F_{k0} + F_{0k})^2 + \frac{1}{2}\sum_{j=1}^{k-1}\{F_{-kj} - F_{kj}, F_{k,-j} - F_{-k,-j}\}, \tag{18}$$

$$D_3 = \frac{\mathbf{i}}{2}\left(F_{k0}^2 - F_{0k}^2\right) + \mathbf{i}\sum_{j=1}^{k-1}(F_{kj}F_{k,-j} - F_{-kj}F_{-k,-j}), \qquad D_0 = -\mathbf{i}F_{kk}.$$

Since the case $k = 1$ does not fit the general scheme due to the triviality of the group K, we assume from now $k \geqslant 2$. The case $k = 1$ will be considered below.

Let the space $\mathcal{R}_{\ell,i}$ equals $V_{\mathfrak{B}_k}(\lambda)$ for a highest weight (A.4), where $m_i \in \mathbb{Z}_+$, and $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ be a subspace of $V_{\mathfrak{B}_k}(\lambda)$ annulled by the subalgebra $\mathfrak{k}^{\mathbb{C}} \cong \mathfrak{B}_{k-1}$. An element $v \in \widetilde{V}_{\mathfrak{B}_k}(\lambda)$, $v \neq 0$, is a highest vector of the trivial one-dimensional $\mathfrak{B}_{k-1}$-module. Then propositions A.1 and A.2 imply the existence of such numbers $m'_j \in \mathbb{Z}_+$, $j = 1, \dots, k$, that

$$\begin{aligned}
&m_k \geqslant m'_k \geqslant m_{k-1} \geqslant \cdots \geqslant m'_2 \geqslant m_1 \geqslant m'_1 \geqslant -m_1,\\
&m'_k \geqslant 0 \geqslant m'_{k-1} \geqslant 0 \geqslant \cdots \geqslant m'_2 \geqslant 0 \geqslant |m'_1|.
\end{aligned}$$

Thus, $m'_j = 0$, $j = 1, \dots, k-1$ and therefore $m_j = 0$, $j = 1, \dots, k-2$.

From now till the end of the present subsection suppose

$$\lambda = m_{k-1}\varepsilon_{k-1} + m_k\varepsilon_k, \qquad m_k \geqslant m_{k-1} \geqslant 0, \qquad m_k, m_{k-1} \in \mathbb{Z}_+.$$

In this case, proposition A.1 implies that every module $V_{\mathfrak{D}_k}(m'_k\varepsilon_k) \subset V_{\mathfrak{B}_k}(\lambda)$ contains the unique one-dimensional module $V_{\mathfrak{B}_{k-1}}(0)$. This fact leads to

$$\dim \widetilde{V}_{\mathfrak{B}_k}(\lambda) = m_k - m_{k-1} + 1. \tag{19}$$

Thus, from proposition 1 one gets the following expansion [42]:

$$\mathcal{L}^2(SO(2k+1), SO(2k-1), \mu) = \bigoplus_{\substack{m_k \geqslant m_{k-1}\\ m_k, m_{k-1} \in \mathbb{Z}_+}} (m_k - m_{k-1} + 1) V_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}),$$

where the left-hand side is considered as a restriction of the left regular representation for the group $SO(2k+1)$. On the other hand, the space

$$\mathcal{L}^2(SO(2k+1), SO(2k-1), \mu)$$

as a $\mathrm{Diff}_{SO(2k+1)}(SO(2k+1)/SO(2k-1))$-module is expanded as

$$\begin{aligned}
&\mathcal{L}^2(SO(2k+1), SO(2k-1), \mu)\\
&\quad = \bigoplus_{\substack{m_k \geqslant m_{k-1}\\ m_k, m_{k-1} \in \mathbb{Z}_+}} \big(\dim V_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})\big) \widetilde{V}_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}),
\end{aligned} \tag{20}$$

where $\dim V_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})$ is given by (A.8).

Let

$$D^+ := \sum_{j=1}^{k-1} F_{kj}F_{k,-j} + \frac{1}{2}F_{k0}^2, \qquad D^- := \sum_{j=1}^{k-1} F_{-kj}F_{-k,-j} + \frac{1}{2}F_{0k}^2,$$

$$\widetilde{C} := C|_{\mathcal{L}^2(SO(2k+1), SO(2k-1), \mu)} = F_{kk}^2 + \{F_{k0}, F_{0k}\} + \sum_{j=1}^{k-1} (\{F_{kj}, F_{jk}\} + \{F_{k,-j}, F_{-jk}\})$$

be operators from $\mathrm{Diff}_{SO(2k+1)}(SO(2k+1)/SO(2k-1))$, where C is the universal Casimir operator (A.6). Due to (A.3) and (A.9), the operator D^+ 'raises' weight subspaces of $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ and the operator D^- 'lowers' them.

Since $[F_{kj}, F_{k,-j}] = [F_{-kj}, F_{-k,-j}] = 0$, one gets the following relations:

$$\begin{aligned}
&D_1 = D^+ + D^- + \frac{1}{2}\big(F_{kk}^2 - \widetilde{C}\big), && D_2 = -D^+ - D^- + \frac{1}{2}\big(F_{kk}^2 - \widetilde{C}\big),\\
&D_3 = \mathbf{i}(D^+ - D^-), && D^+ = \frac{1}{4}(D_1 - D_2) - \frac{\mathbf{i}}{2}D_3,\\
&D^- = \frac{1}{4}(D_1 - D_2) + \frac{\mathbf{i}}{2}D_3, && \widetilde{C} = -D_0^2 - D_1 - D_2.
\end{aligned} \tag{21}$$

Commutator relations (8) now give

$$[F_{kk}, D^+] = 2D^+, \qquad [F_{kk}, D^-] = -2D^-, \tag{22}$$

$$[D^+, D^-] = -\tfrac{1}{2}F_{kk}^3 + \tfrac{1}{2}\widetilde{C}F_{kk} + \tfrac{1}{4}(2k-1)(2k-3)F_{kk}. \tag{23}$$

Formulae (A.5) and (A.7) imply

$$\widetilde{C}|_{\widetilde{V}_{\mathfrak{B}_k}(\lambda)} = \left(\left(k + m_k - \frac{1}{2}\right)^2 + \left(k + m_{k-1} - \frac{3}{2}\right)^2 - \left(k - \frac{1}{2}\right)^2 - \left(k - \frac{3}{2}\right)^2\right) \mathrm{id}. \tag{24}$$

It follows from the paper [36] that

$$\widetilde{V}_{\mathfrak{B}_k}(\lambda) = V_{-\nu\varepsilon_k} \oplus V_{-(\nu-2)\varepsilon_k} \oplus \cdots \oplus V_{(\nu-2)\varepsilon_k} \oplus V_{\nu\varepsilon_k}, \tag{25}$$

where $\nu = m_k - m_{k-1}$ and all summands are one-dimensional weight spaces w.r.t. the Cartan subalgebra $\mathfrak{h}_k$. Formulae (A.3) and (A.9) imply

$$D^+ \colon V_{j\varepsilon_k} \to V_{(j+2)\varepsilon_k}, \qquad D^- \colon V_{j\varepsilon_k} \to V_{(j-2)\varepsilon_k}.$$

The action of operators F_{kk}, D^+, D^- in the space $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ was calculated in [36] w.r.t. some base. In particular, in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ there are no nontrivial invariant subspaces w.r.t. this action. We shall obtain simpler formulae for the D^+- and D^--action w.r.t. a base in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ with a normalization different from those in [36].

Lemma 1. *Let $L_\nu := (-\nu, -\nu+2, \ldots, \nu-2, \nu)$. There is a base $(\chi_j)_{j\in L_\nu}$ in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$ such that*

$$F_{kk}\chi_j = j\chi_j, \qquad D^+\chi_j = \tfrac{1}{4}(j - m_k - m_{k-1} - 2k + 3)(j - \nu)\chi_{j+2}, \tag{26}$$

$$D^-\chi_j = \tfrac{1}{4}(j + m_k + m_{k-1} + 2k - 3)(j + \nu)\chi_{j-2}, \tag{27}$$

where $\chi_j = 0$ if $j \notin L_\nu$.

Proof. Since the action of an algebra, generated by operators F_{kk}, D^+, D^-, is irreducible in $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$, one can define by induction nonzero elements $\chi_j \in V_{j\varepsilon_k}$, $j \in L_\nu$ such that formulae (26) are valid. Prove by induction formula (27). For $j = -\nu$ it is evident. Suppose that (27) is valid for $j = -\nu, -\nu+2, \ldots, i$, where $i < \nu$. Then using (24) one gets

$$\begin{aligned}
\tfrac{1}{4}(i - m_k - m_{k-1} - 2k + 3)(i - \nu)D^-\chi_{i+2} &= D^-D^+\chi_i = ([D^-, D^+] + D^+D^-)\chi_i \\
&= \left(\tfrac{1}{2}F_{kk}^3 - \tfrac{1}{2}\widetilde{C}F_{kk} - \tfrac{1}{4}(2k-1)(2k-3)F_{kk}\right)\chi_i \\
&\quad + \tfrac{1}{4}(i + m_k + m_{k-1} + 2k - 3)(i + \nu)D^+\chi_{i-2} \\
&= \tfrac{1}{2}\left(i^3 - i\left(m_k^2 + m_{k-1}^2 + (2k-1)m_k + (2k-3)m_{k-1} + \tfrac{1}{2}(2k-1)(2k-3)\right)\right)\chi_i \\
&\quad + \tfrac{1}{16}(i + m_k + m_{k-1} + 2k - 3)(i + \nu)(i - m_k - m_{k-1} - 2k + 1)(i - 2 - \nu)\chi_i \\
&= \tfrac{1}{16}(i - m_k - m_{k-1} - 2k + 3)(i - \nu)(i + m_k + m_{k-1} + 2k - 1)(i + 2 + \nu)\chi_i,
\end{aligned}$$

due to the identity

$$\begin{aligned}
&(i - m_k - m_{k-1} - 2k + 3)(i - \nu)(i + m_k + m_{k-1} + 2k - 1)(i + 2 + \nu) \\
&\quad - (i + m_k + m_{k-1} + 2k - 3)(i + \nu)(i - m_k - m_{k-1} - 2k + 1)(i - 2 - \nu) \\
&= 8i^3 - 8i\left(m_k^2 + m_{k-1}^2 + (2k-1)m_k + (2k-3)m_{k-1} + \tfrac{1}{2}(2k-1)(2k-3)\right).
\end{aligned}$$

Since $(i - m_k - m_{k-1} - 2k + 3)(i - \nu) \neq 0$, we obtain

$$D^-\chi_{i+2} = \tfrac{1}{4}(i + m_k + m_{k-1} + 2k - 1)(i + 2 + \nu)\chi_i$$

that completes the induction. □

Lemma 1, expansion (20) and relations (21) effectively describe the action of operators D_0, D_1, D_2, D_3 in the space $\mathcal{L}^2(SO(2k+1), SO(2k-1), \mu)$. Consider the problem of finding all common eigenvectors ψ_D of operators D_0^2, D_1, D_2 and optionally D_3. It is equivalent to the problem of finding all common eigenvectors of operators $D_0^2, D^+ + D^-$ and optionally $D^+ - D^-$ in the space $\widetilde{V}_{\mathfrak{B}_k}(\lambda)$.

Eigenvectors for the operator D_0^2 are

$$c_+\chi_j + c_-\chi_{-j}, c_\pm \in \mathbb{C}, j \in L_\nu, j \geqslant 0$$

with eigenvalues $-j^2$. Since

$$(D^+ + D^-)(c_+\chi_j + c_-\chi_{-j}) = \tfrac{1}{4}(j - m_k - m_{k-1} - 2k+3)(j-\nu)(c_+\chi_{j+2} + c_-\chi_{-j-2}) + \tfrac{1}{4}(j + m_k + m_{k-1} + 2k-3)(j+\nu)(c_+\chi_{j-2} + c_-\chi_{-j+2}),$$

the requirement

$$(D^+ + D^-)(c_+\chi_j + c_-\chi_{-j}) \sim c_+\chi_j + c_-\chi_{-j}$$

implies $(j - m_k - m_{k-1} - 2k+3)(j-\nu) = 0$ that leads to two cases: $j = m_k - m_{k-1}$ and $j = m_k + m_{k-1} + 2k - 3$.

In the first case, one gets

$$(D^+ + D^-)(c_+\chi_{m_k - m_{k-1}} + c_-\chi_{-m_k+m_{k-1}}) = (m_k - m_{k-1})\left(m_k + k - \frac{3}{2}\right)(c_+\chi_{m_k-m_{k-1}-2} + c_-\chi_{-m_k+m_{k-1}+2})$$

that implies one of three possibilities

1. $m_k - m_{k-1} = 0$;
2. $m_k - m_{k-1} - 2 = -m_k + m_{k-1}$;
3. $m_k - m_{k-1} - 2 = 0, c_+ + c_- = 0$.

Thus, we obtain the following eigenvectors:

1. $(D^+ + D^-)\chi_0 = 0$ for $m_k - m_{k-1} = 0$;
2. $(D^+ + D^-)(\chi_1 + \chi_{-1}) = \left(m_k + k - \frac{3}{2}\right)(\chi_1 + \chi_{-1})$ for $m_k \in \mathbb{N}, m_{k-1} = m_k - 1$;
3. $(D^+ + D^-)(\chi_1 - \chi_{-1}) = -\left(m_k + k - \frac{3}{2}\right)(\chi_1 - \chi_{-1})$ for $m_k \in \mathbb{N}, m_{k-1} = m_k - 1$;
4. $(D^+ + D^-)(\chi_2 - \chi_{-2}) = 0, m_{k-1} = m_k - 2, m_k = 2, 3, \ldots$.

In the second case, one gets $m_k + m_{k-1} + 2k - 3 = j \leqslant m_k - m_{k-1}$ that implies $0 \leqslant m_{k-1} \leqslant \frac{3}{2} - k$ and thus $k = 1$ that contradicts to the assumption $k \geqslant 2$.

Using relations (21) this consideration can be summarized in the following proposition.

Proposition 3. *For $n = 2k, k \geqslant 2$ there are four series of common eigenvectors in $\widetilde{V}_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1}), m_k, m_{k-1} \in \mathbb{Z}_+$ for the operators D_0^2, D_1, D_2:*

1. $D_0^2\chi_0 = D_3\chi_0 = 0, D_1\chi_0 = D_2\chi_0 = -m_k(m_k + 2k - 2)\chi_0, m_k = m_{k-1}$;
2. $D_0^2(\chi_1 + \chi_{-1}) = -(\chi_1 + \chi_{-1}), D_2(\chi_1 + \chi_{-1}) = -m_k(m_k + 2k-2)(\chi_1 + \chi_{-1})$,
   $D_1(\chi_1 + \chi_{-1}) = \left(-m_k^2 - 2(k-2)m_k + 2k - 3\right)(\chi_1 + \chi_{-1})$,
   $D_3(\chi_1 + \chi_{-1}) = \mathbf{i}\left(m_k + k - \frac{3}{2}\right)(\chi_1 - \chi_{-1}), m_{k-1} = m_k - 1, m_k \in \mathbb{N}$
3. $D_0^2(\chi_1 - \chi_{-1}) = -(\chi_1 - \chi_{-1}), D_1(\chi_1 - \chi_{-1}) = -m_k(m_k + 2k-2)(\chi_1 - \chi_{-1})$,
   $D_2(\chi_1 - \chi_{-1}) = \left(-m_k^2 - 2(k-2)m_k + 2k - 3\right)(\chi_1 - \chi_{-1})$,
   $D_3(\chi_1 - \chi_{-1}) = -\mathbf{i}\left(m_k + k - \frac{3}{2}\right)(\chi_1 + \chi_{-1}), m_{k-1} = m_k - 1, m_k \in \mathbb{N}$*;*
4. $D_0^2(\chi_2 - \chi_{-2}) = -4(\chi_2 - \chi_{-2}), D_3(\chi_2 - \chi_{-2}) = -4\mathbf{i}\left(m_k + k - \frac{3}{2}\right)\chi_0$,
   $D_1(\chi_2 - \chi_{-2}) = D_2(\chi_2 - \chi_{-2}) = \left(-m_k^2 - 2(k-2)m_k + 2k - 3\right)(\chi_2 - \chi_{-2})$,
   $m_{k-1} = m_k - 2, m_k = 2, 3, 4, \ldots$.

Only the first vector is also an eigenvector for the operator D_3.

Multiplicities of corresponding eigenvalues in $\mathcal{L}^2(SO(n+1), SO(n-1), \mu)$ *are equal to* $\dim V_{\mathfrak{B}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})$ *and can be calculated in explicit form using (A.8).*

Consider the case $k = 1, n = 2$. Now the group K is trivial and therefore $\widetilde{V}_{\mathfrak{B}_1}(\lambda) = V_{\mathfrak{B}_1}(\lambda)$. The algebra $\mathfrak{B}_1 = \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{A}_1$ is spanned by elements F_{11}, F_{01}, F_{10} with commutator relations

$$[F_{11}, F_{01}] = -F_{01}, \qquad [F_{11}, F_{10}] = F_{10}, \qquad [F_{10}, F_{01}] = F_{11}.$$

Its representation theory is well known: all its finite-dimensional irreducible modules are of the form

$$V_{\mathfrak{B}_1}(m\varepsilon_1) = V_{-m\varepsilon_1} \oplus V_{-(m-1)\varepsilon_1} \oplus \cdots \oplus V_{(m-1)\varepsilon_1} \oplus V_{m\varepsilon_1},$$

where $m \in \mathbb{Z}_+ \cup \left(\mathbb{Z}_+ + \frac{1}{2}\right)$, all $V_{j\varepsilon_1}$ are one-dimensional weight subspaces w.r.t. $\mathfrak{h}_1 = \mathrm{span}(F_{11})$ and

$$\begin{aligned} F_{10}&\colon \quad V_{j\varepsilon_1} \to V_{(j+1)\varepsilon_1}, \qquad & j = -m, \ldots, m-1,\\ F_{01}&\colon \quad V_{j\varepsilon_1} \to V_{(j-1)\varepsilon_1}, \qquad & j = -m+1, \ldots, m \end{aligned}$$

are bijections.

We shall consider only $m \in \mathbb{Z}_+$ since

$$\mathcal{L}^2(SO(3), \mu) = \bigoplus_{m\in\mathbb{Z}_+} (2m+1) V_{\mathfrak{B}_1}(m\varepsilon_1).$$

Thus, there are additional weight subspaces in the module $V_{\mathfrak{B}_1}(m\varepsilon_1)$ w.r.t. expansion (25) and the action of the algebra, generated by the operators $D^+ = \frac{1}{2}F_{10}^2$, $D^- = \frac{1}{2}F_{01}^2$, is not irreducible in $V_{\mathfrak{B}_1}(m\varepsilon_1)$.

One can choose a base $(\chi_j)_{j=-m}^m$ in $V_{\mathfrak{B}_1}(m\varepsilon_1)$ such that

$$\chi_j \in V_{j\varepsilon_1}, \qquad F_{11}\chi_j = j\chi_j, \qquad F_{10}\chi_j = -\tfrac{1}{\sqrt{2}}\sqrt{(m-j)(m+j+1)}\,\chi_{j+1},$$
$$F_{01}\chi_j = -\tfrac{1}{\sqrt{2}}\sqrt{(m+j)(m-j+1)}\,\chi_{j-1},$$

where as above $\chi_j = 0$ for $|j| > m$.

Eigenvectors for the operator $D_0^2 = -F_{11}^2$ are

$$c_+\chi_j + c_-\chi_{-j}, c_\pm \in \mathbb{C}, j = 0, 1, \ldots, m$$

with eigenvalues $-j^2$. Since

$$\begin{aligned}(D^+ + D^-)&(c_+\chi_j + c_-\chi_{-j})\\ &= \tfrac{1}{4}\sqrt{(m-j)(m+j+1)(m-j-1)(m+j+2)}(c_+\chi_{j+2} + c_-\chi_{-j-2})\\ &\quad + \tfrac{1}{4}\sqrt{(m+j)(m-j+1)(m+j-1)(m-j+2)}(c_+\chi_{j-2} + c_-\chi_{-j+2}),\end{aligned}$$

the requirement

$$(D^+ + D^-)(c_+\chi_j + c_-\chi_{-j}) \sim c_+\chi_j + c_-\chi_{-j}$$

implies $(m-j)(m+j+1)(m-j-1)(m+j+2) = 0$ that gives two cases: $j = m$ and $j = m-1$.

In the first case, one gets

$$(D^+ + D^-)(c_+\chi_m + c_-\chi_{-m}) = \tfrac{1}{2}\sqrt{m(2m-1)}(c_+\chi_{m-2} + c_-\chi_{m+2})$$

that implies one of three possibilities

1. $m = j = 0$;
2. $m - 2 = -m$;
3. $m - 2 = 0, c_+ + c_- = 0$.

This gives the following eigenvectors:

1. $(D^+ + D^-)\chi_0 = 0, m = 0$;
2. $(D^+ + D^-)(\chi_1 + \chi_{-1}) = \frac{1}{2}(\chi_1 + \chi_{-1}), m = 1$;
3. $(D^+ + D^-)(\chi_1 - \chi_{-1}) = -\frac{1}{2}(\chi_1 - \chi_{-1}), m = 1$;
4. $(D^+ + D^-)(\chi_2 - \chi_{-2}) = 0, m = 2$.

It is easily seen that these eigenvectors correspond to eigenvectors from proposition 3 for $m_k = m, m_{k-1} = 0$.

In the second case, it holds

$$(D^+ + D^-)(c_+\chi_{m-1} + c_-\chi_{-m+1}) = \tfrac{1}{2}\sqrt{3(2m-1)(m-1)}(c_+\chi_{m-3} + c_-\chi_{m+3})$$

that implies one of three possibilities

1. $m = 1, j = 0$;
2. $m - 3 = -m + 1$;
3. $m - 3 = 0, c_+ + c_- = 0$.

Thus, one gets the following eigenvectors:

1. $(D^+ + D^-)\chi_0 = 0, m = 1$;
2. $(D^+ + D^-)(\chi_1 + \chi_{-1}) = \frac{3}{2}(\chi_1 + \chi_{-1}), m = 2$;
3. $(D^+ + D^-)(\chi_1 - \chi_{-1}) = -\frac{3}{2}(\chi_1 - \chi_{-1}), m = 2$;
4. $(D^+ + D^-)(\chi_2 - \chi_{-2}) = 0, m = 3$.

Since it holds $\widetilde{C}\big|_{\widetilde{V}_{\mathfrak{B}_1}(m\varepsilon_1)} = m(m+1)\,\mathrm{id}$ and relations (21) are also valid in the case $k = 1$ one gets the following proposition.

Proposition 4. *There are eight common eigenvectors in $V_{\mathfrak{B}_1}(m\varepsilon_1)$ for the operators D_0^2, D_1, D_2:*

1. $D_0^2\chi_0 = D_1\chi_0 = D_2\chi_0 = D_3\chi_0 = 0, m = 0$;
2. $D_0^2\chi_0 = D_3\chi_0 = 0, D_1\chi_0 = D_2\chi_0 = -\chi_0, m = 1$;
3. $D_0^2(\chi_1 + \chi_{-1}) = D_2(\chi_1 + \chi_{-1}) = -(\chi_1 + \chi_{-1}), D_1(\chi_1 + \chi_{-1}) = 0,$
   $D_3(\chi_1 + \chi_{-1}) = \dfrac{\mathbf{i}}{2}(\chi_1 - \chi_{-1}), m = 1$;
4. $D_0^2(\chi_1 - \chi_{-1}) = D_1(\chi_1 - \chi_{-1}) = -(\chi_1 - \chi_{-1}), D_2(\chi_1 - \chi_{-1}) = 0,$
   $D_3(\chi_1 - \chi_{-1}) = -\dfrac{\mathbf{i}}{2}(\chi_1 + \chi_{-1}), m = 1$;
5. $D_0^2(\chi_2 - \chi_{-2}) = -4(\chi_2 - \chi_{-2}), D_1(\chi_2 - \chi_{-2}) = D_2(\chi_2 - \chi_{-2}) = -(\chi_2 - \chi_{-2}),$
   $D_3(\chi_2 - \chi_{-2}) = -\sqrt{6}\mathbf{i}\chi_0, m = 2$;
6. $D_0^2(\chi_1 + \chi_{-1}) = D_1(\chi_1 + \chi_{-1}) = -(\chi_1 + \chi_{-1}), D_2(\chi_1 + \chi_{-1}) = -4(\chi_1 + \chi_{-1}),$
   $D_3(\chi_1 + \chi_{-1}) = \dfrac{3}{2}\mathbf{i}(\chi_1 - \chi_{-1}), m = 2$;
7. $D_0^2(\chi_1 - \chi_{-1}) = D_2(\chi_1 - \chi_{-1}) = -(\chi_1 - \chi_{-1}), D_1(\chi_1 - \chi_{-1}) = -4(\chi_1 - \chi_{-1}),$
   $D_3(\chi_1 - \chi_{-1}) = -\dfrac{3}{2}\mathbf{i}(\chi_1 + \chi_{-1}), m = 2$;

8. $D_0^2(\chi_2-\chi_{-2})=D_1(\chi_2-\chi_{-2})=D_2(\chi_2-\chi_{-2})=-4(\chi_2-\chi_{-2})$,
$D_3(\chi_2-\chi_{-2})=-\sqrt{30}\,\mathbf{i}\chi_0$, $m=3$.

Only the first and the second vectors are also the eigenvectors for the operator D_3.
Multiplicities of corresponding eigenvalues in $\mathcal{L}^2(SO(3),\mu)$ are $2m+1$.

5.2. The case $n=2k-1$

Here, we use notation from appendix A.2. The algebra $\mathfrak{D}_k$ is considered there as a subalgebra of $\mathfrak{B}_k$. Therefore, one can easily obtain analogues of formulae (18) simply by deleting the terms F_{k0} and F_{0k}:

$$D_0=-\mathbf{i}F_{kk},\qquad D_1=\frac{1}{2}\sum_{j=1}^{k-1}\{F_{-kj}+F_{kj},F_{k,-j}+F_{-k,-j}\},$$

$$D_2=\frac{1}{2}\sum_{j=1}^{k-1}\{F_{-kj}-F_{kj},F_{k,-j}-F_{-k,-j}\},\qquad D_3=\mathbf{i}\sum_{j=1}^{k-1}(F_{kj}F_{k,-j}-F_{-kj}F_{-k,-j}).$$

Let the space $\mathcal{R}_{\ell,i}$ equals $V_{\mathfrak{D}_k}(\lambda)$ for a highest weight (A.10), where $m_i\in\mathbb{Z}_+$, $i\geqslant 2$, $m_1\in\mathbb{Z}$, and $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ be a subspace of $V_{\mathfrak{D}_k}(\lambda)$ annulled by the subalgebra $\mathfrak{k}^{\mathbb{C}}\cong\mathfrak{D}_{k-1}$. Reasoning as above in the case $n=2k$, one gets that $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ is nontrivial iff

$$\lambda=m_k\varepsilon_k+m_{k-1}\varepsilon_{k-1},\qquad m_k\geqslant|m_{k-1}|,\qquad m_k\in\mathbb{Z}_+,\qquad m_{k-1}\in\mathbb{Z}'_k,\tag{28}$$

where $\mathbb{Z}'_k=\mathbb{Z}_+$ for $k\geqslant 3$ and $\mathbb{Z}'_2=\mathbb{Z}$. In this case, one has $\dim\widetilde{V}_{\mathfrak{D}_k}(\lambda)=m_k-|m_{k-1}|+1$.

Below in the present subsection we suppose that condition (28) is valid. This leads to the expansion

$$\mathcal{L}^2(SO(2k),SO(2k-2),\mu)=\bigoplus_{\substack{m_k\geqslant|m_{k-1}|\\ m_k\in\mathbb{Z}_+,m_{k-1}\in\mathbb{Z}'_k}}(m_k-|m_{k-1}|+1)V_{\mathfrak{D}_k}(m_k\varepsilon_k+m_{k-1}\varepsilon_{k-1})$$

of the left $SO(2k)$-space $\mathcal{L}^2(SO(2k),SO(2k-2),\mu)$ and to the expansion

$$\mathcal{L}^2(SO(2k),SO(2k-2),\mu)=\bigoplus_{\substack{m_k\geqslant|m_{k-1}|\\ m_k\in\mathbb{Z}_+,m_{k-1}\in\mathbb{Z}'_k}}\big(\dim V_{\mathfrak{D}_k}(m_k\varepsilon_k+m_{k-1}\varepsilon_{k-1})\big)\widetilde{V}_{\mathfrak{D}_k}(m_k\varepsilon_k+m_{k-1}\varepsilon_{k-1}),\tag{29}$$

of the same space as a $\mathrm{Diff}_{SO(2k)}(SO(2k)/SO(2k-2))$-module, where the dimension $\dim V_{\mathfrak{D}_k}(m_k\varepsilon_k+m_{k-1}\varepsilon_{k-1})$ is given by (A.8).

Now let

$$D^+:=\sum_{j=1}^{k-1}F_{kj}F_{k,-j},\qquad D^-:=\sum_{j=1}^{k-1}F_{-kj}F_{-k,-j},$$

$$\widetilde{C}:=C|_{\mathcal{L}^2(SO(2k),SO(2k-2),\mu)}=F_{kk}^2+\sum_{j=1}^{k-1}(\{F_{kj},F_{jk}\}+\{F_{k,-j},F_{-jk}\})$$

be operators from $\mathrm{Diff}_{SO(2k)}(SO(2k)/SO(2k-2))$, where C is the universal Casimir operator (A.11).

Formulae (21) and (22) are valid without any modification and formula (23) becomes

$$[D^+, D^-] = -\tfrac{1}{2}F_{kk}^3 + \tfrac{1}{2}\widetilde{C}F_{kk} + (k-1)(k-2)F_{kk}.$$

Now

$$\widetilde{C}|_{\widetilde{V}_{\mathfrak{D}_k}(\lambda)} = ((m_k+k-1)^2 + (m_{k-1}+k-2)^2 - (k-1)^2 - (k-2)^2)\,\mathrm{id}.$$

From [35] it follows that

$$\widetilde{V}_{\mathfrak{D}_k}(\lambda) = V_{-\nu\varepsilon_k} \oplus V_{-(\nu-2)\varepsilon_k} \oplus \cdots \oplus V_{(\nu-2)\varepsilon_k} \oplus V_{\nu\varepsilon_k},$$

where $\nu = m_k - |m_{k-1}|$, all summands are one-dimensional weight spaces w.r.t. the Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{D}_k$ and the algebra, generated by the operators D^+, D^-, acts in $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ in an irreducible way.

Again we shall simplify formulae for this action w.r.t. [35] using another base. The next lemma can be proved completely similar to the proof of lemma 1.

Lemma 2. *Let $\nu := m_k - |m_{k-1}|$, $L_\nu := (-\nu, -\nu+2, \ldots, \nu-2, \nu)$. There is a base $(\chi_j)_{j\in L_\nu}$ in $\widetilde{V}_{\mathfrak{D}_k}(\lambda)$ such that*

$$F_{kk}\chi_j = j\chi_j, \qquad D^+\chi_j = \tfrac{1}{4}(j - m_k - |m_{k-1}| - 2k + 4)(j-\nu)\chi_{j+2},$$
$$D^-\chi_j = \tfrac{1}{4}(j + m_k + |m_{k-1}| + 2k - 4)(j+\nu)\chi_{j-2},$$

where $\chi_j = 0$ if $j \notin L_\nu$.

Arguing as in the $\mathfrak{B}_k$-case one gets the following proposition.

Proposition 5. *For $n = 2k-1, k \geqslant 2$, there are four series of common eigenvectors in $\widetilde{V}_{\mathfrak{D}_k}(m_k\varepsilon_k + |m_{k-1}|\varepsilon_{k-1})$, $m_k \in \mathbb{Z}_+$, $m_{k-1} \in \mathbb{Z}'_k$ for the operators D_0^2, D_1, D_2:*

1. $D_0^2\chi_0 = D_3\chi_0 = 0$, $D_1\chi_0 = D_2\chi_0 = -m_k(m_k+2k-3)\chi_0$, $m_k = |m_{k-1}|$;
2. $D_0^2(\chi_1+\chi_{-1}) = -(\chi_1+\chi_{-1})$, $D_2(\chi_1+\chi_{-1}) = -m_k(m_k+2k-3)(\chi_1+\chi_{-1})$,
   $D_1(\chi_1+\chi_{-1}) = \big(-m_k^2 + (5-2k)m_k + 2k - 4\big)(\chi_1+\chi_{-1})$,
   $D_3(\chi_1+\chi_{-1}) = \mathbf{i}(m_k+k-2)(\chi_1-\chi_{-1})$, $|m_{k-1}| = m_k - 1$, $m_k \in \mathbb{N}$
3. $D_0^2(\chi_1-\chi_{-1}) = -(\chi_1-\chi_{-1})$, $D_1(\chi_1-\chi_{-1}) = -m_k(m_k+2k-3)(\chi_1-\chi_{-1})$,
   $D_2(\chi_1-\chi_{-1}) = \big(-m_k^2 + (5-2k)m_k + 2k - 4\big)(\chi_1-\chi_{-1})$,
   $D_3(\chi_1-\chi_{-1}) = -\mathbf{i}(m_k+k-2)(\chi_1+\chi_{-1})$, $|m_{k-1}| = m_k - 1$, $m_k \in \mathbb{N}$;
4. $D_0^2(\chi_2-\chi_{-2}) = -4(\chi_2-\chi_{-2})$, $D_3(\chi_2-\chi_{-2}) = -4\mathbf{i}(m_k+k-2)\chi_0$,
   $D_1(\chi_2-\chi_{-2}) = D_2(\chi_2-\chi_{-2}) = \big(-m_k^2 + (5-2k)m_k + 2k - 4\big)(\chi_2-\chi_{-2})$,
   $|m_{k-1}| = m_k - 2$, $m_k = 2, 3, 4, \ldots$.

Only the first vector is also an eigenvector for the operator D_3.

Multiplicities of corresponding eigenvalues in $\mathcal{L}^2(SO(n+1), SO(n-1), \mu)$ are equal to $\dim V_{\mathfrak{D}_k}(m_k\varepsilon_k + m_{k-1}\varepsilon_{k-1})$ *and can be calculated in explicit form using (A.8).*

Remark 1. For $k = 2$, a value of $m_{k-1} = m_1$ can have an arbitrary sign and one gets eight common eigenvectors found in [34].

Remark 2. Results of propositions 3–5 correspond to proposition 2 and are even more restrictive. Indeed, if $\psi_D \in \mathcal{L}^2(SO(n+1), SO(n-1), \mu)$ is an eigenfunction for operators D_0^2, D_1, D_2 and D_3, then $D_0\psi_D = D_3\psi_D = 0$, $D_1\psi_D = D_2\psi_D$.

6. Scalar spectral equations and some energy levels for the two-body problem in S^n

Here, we shall consider the spectral problem (13), where the operator H is defined in (11) and ψ_D is one of common eigenfunctions for operators D_0^2, D_1, D_2 and optionally D_3. In this section, m denotes reduced mass (10) and integers m_k correspond to highest weights in $\mathfrak{so}(n+1)$-modules.

Let $D_0^2\psi_D = \delta_0\psi_D$, $D_i\psi_D = \delta_i\psi_D$, $i = 1, 2$. In accordance with remark 2 there are two main cases:

1. $D_3\psi_D = 0$, $\delta_0 = 0$, $\delta_1 = \delta_2$, particle masses are arbitrary;
2. $D_3\psi_D \not\sim \psi_D$, particle masses are equal.

In the first case,

$$(CD_1 + AD_2 + 2BD_3)\psi_D = \delta_1(C + A)\psi_D = \frac{(1+r^2)^2}{4mR^2r^2}\delta_1\psi_D.$$

In the second case,

$$A = \frac{1+r^2}{4mR^2r^2}, \qquad B \equiv 0, \qquad C = \frac{1+r^2}{4mR^2}.$$

In all cases, one gets the following spectral equation for the function $f(r)$:

$$f'' + \frac{n-1+(3-n)r^2}{(1+r^2)r}f' + \frac{8}{(1+r^2)^2}\left(mR^2(E - V(r)) - \frac{a}{r^2} - b - cr^2\right)f = 0,$$
$$a, b, c \geqslant 0, \qquad 0 < r < \infty. \tag{30}$$

where coefficients a, b, c are described below.

For eigenfunctions ψ_D classified in proposition 3 ($n = 2k$, $k = 2, 3, \ldots$), one has

1. $a = c = m_k(m_k + 2k - 2)/8$, $b = 2a$, $m_k \in \mathbb{Z}_+$, masses are arbitrary;
2. $a = m_k(m_k + 2k - 2)/8$, $b = \left(m_k^2 + (2k-3)m_k - k + 2\right)/4$, $c = \left(m_k^2 + 2(k-2)m_k - 2k + 3\right)/8$, $m_k \in \mathbb{N}$, masses are equal;
3. $a = \left(m_k^2 + 2(k-2)m_k - 2k + 3\right)/8$, $b = \left(m_k^2 + (2k-3)m_k - k + 2\right)/4$, $c = m_k(m_k + 2k - 2)/8$, $m_k \in \mathbb{N}$, masses are equal;
4. $a = c = \left(m_k^2 + 2(k-2)m_k - 2k + 3\right)/8$, $b = \left(m_k^2 + 2(k-2)m_k - 2k + 5\right)/4$, $m_k = 2, 3, \ldots$, masses are equal.

Proposition 4 ($n = 2$) gives the following values for a, b, c:

1. $a = c = b = 0$, masses are arbitrary;
2. $a = c = 1/8$, $b = 1/4$, masses are arbitrary;
3. $a = 1/8$, $b = 1/4$, $c = 0$, masses are equal;
4. $a = 0$, $b = 1/4$, $c = 1/8$, masses are equal;
5. $a = c = 1/8$, $b = 3/4$, masses are equal;
6. $a = 1/2$, $b = 3/4$, $c = 1/8$, masses are equal;
7. $a = 1/8$, $b = 3/4$, $c = 1/2$, masses are equal;
8. $a = c = 1/2$, $b = 3/2$, masses are equal.

Finally, proposition 5 corresponds to the following cases ($n = 2k - 1$, $k = 2, 3, \ldots$):

1. $a = c = m_k(m_k + 2k - 3)/8$, $b = 2a$, $m_k \in \mathbb{Z}_+$, masses are arbitrary;
2. $a = m_k(m_k + 2k - 3)/8$, $b = \left(m_k^2 + (2k-4)m_k - k + \frac{5}{2}\right)/4$, $c = \left(m_k^2 + (2k-5)m_k - 2k + 4\right)/8$, $m_k \in \mathbb{N}$, masses are equal;
3. $a = \left(m_k^2 + (2k-5)m_k - 2k + 4\right)/8$, $b = \left(m_k^2 + (2k-4)m_k - k + \frac{5}{2}\right)/4$, $c = m_k(m_k + 2k - 3)/8$, $m_k \in \mathbb{N}$, masses are equal;

4. $a = c = \left(m_k^2 + (2k-5)m_k - 2k+4\right)/8$, $b = \left(m_k^2 + (2k-5)m_k - 2k+6\right)/4$, $m_k = 2, 3, \ldots,$ masses are equal.

We shall consider equation (30) for the Coulomb and oscillator potentials.

6.1. Coulomb potential

For the Coulomb potential,

$$V_c = -\frac{\gamma}{R}\cot\frac{\rho}{R} = \frac{\gamma}{2R}\left(r - \frac{1}{r}\right), \qquad \gamma > 0 \tag{31}$$

theorems 1 and B.1 imply the self-adjointness of the two-body Hamiltonian H_{V_c} with its domain defined by (B.1), where $V_1 = 0$ for $0 < r < 1$ and $V_1 = V_c$ for $1 \leqslant r < \infty$.

Equation (30) for $V = V_c$ is the Fuchsian differential equation (see appendix C) with four singular points $r = 0, \pm\mathbf{i}, \infty$ and corresponding characteristic exponents:

$$\begin{aligned}
&\rho_\pm^{(0)} = \tfrac{1}{2}\left(2 - n \pm \sqrt{(n-2)^2 + 32a}\right), \qquad \rho_\pm^{(\infty)} = \tfrac{1}{2}\left(2 - n \pm \sqrt{(n-2)^2 + 32c}\right),\\
&\rho_\pm^{(\mathbf{i})} = \tfrac{1}{2}\left(n - 1 \pm \sqrt{(n-1)^2 + 8(mER^2 - \mathbf{i}mR\gamma + a - b + c)}\right),\\
&\rho_\pm^{(-\mathbf{i})} = \tfrac{1}{2}\left(n - 1 \pm \sqrt{(n-1)^2 + 8(mER^2 + \mathbf{i}mR\gamma + a - b + c)}\right).
\end{aligned} \tag{32}$$

Here and below we suppose that a square root for a positive number is positive; for other numbers it is an arbitrary root.

The requirement $f(r)\psi_D \in \mathrm{Dom}(H_{V_c})$ restricts asymptotics of $f(r)$ near singular points $r = 0$ and $r = \infty$. Let $f(r) \sim r^{\rho^{(0)}}$ as $r \to +0$ and $f(r) \sim r^{-\rho^{(\infty)}}$ as $r \to +\infty$. We shall show that $f(r)\psi_D \in \mathrm{Dom}(H_{V_c})$ iff $\rho^{(0)} = \rho_+^{(0)}$ and $\rho^{(\infty)} = \rho_+^{(\infty)}$.

The inclusion

$$f \in \mathcal{L}^2\left(\mathbb{R}_+, \frac{r^{n-1}\,\mathrm{d}r}{(1+r^2)^n}\right)$$

evidently implies $\rho^{(0)} > -n/2$, $\rho^{(\infty)} > -n/2$. On the other hand, one can easily see that the inequality $a \geqslant 1/8$ leads to $\rho_-^{(0)} \leqslant -n/2$ and the inequality $c \geqslant 1/8$ leads to $\rho_-^{(\infty)} \leqslant -n/2$.

From the consideration above it follows that if $a < 1/8$ $(c < 1/8)$ then $a = 0$ $(c = 0)$. For $a = 0$, the inequality $\rho_-^{(0)} = 2 - n > -n/2$ implies $n < 4$.

For $a = 0, n = 3$, the asymptotic $f(r) \sim r^{\rho_-^{(0)}} = 1/r$ means that $\triangle(f\psi_D) \sim \delta(0)$ as $r \to 0$ that contradicts to

$$\triangle\,(f\psi_D) \in \mathcal{L}^2_{\mathrm{loc}}(\mathbf{S}^n \times \mathbf{S}^n, \chi \times \chi), \tag{33}$$

see theorem B.1.

For the case $a = 0, n = 2$, $\rho_+^{(0)} = \rho_-^{(0)} = 0$ holds and the theory of Fuchsian differential equations [55, 56] implies that canonical asymptotics of a solution for (30) near $r = 0$ are 1 and $\log r$. The latter asymptotic again leads to $\triangle(f\psi_D) \sim \delta(0)$ as $r \to 0$ that again contradicts to (33).

Thus, in all cases it should be $f(r) \sim r^{\rho_+^{(0)}}$ as $r \to 0$. Reasoning in a similar way one also gets in all cases the asymptotic $f(r) \sim r^{-\rho_+^{(\infty)}}$ as $r \to +\infty$.

Consider the problem of reducing equation (30) with potential (31) to the hypergeometric equation via reducing (30) to the Heun equation by transformations (C.2), (C.3) and then using theorem C.1.

Singular points of equation (30) form a harmonic quadruple (see appendix C). Therefore, one can use only the first case of theorem C.1. Move singular points $(0, \pm\mathbf{i}, \infty)$ of equation (30) to the quadruple $(0, 1, 2, \infty)$ by a fractional linear transformation $t = \tau(r)$ of the independent variable.

Since the order of singular points on a circle or on a line is conserved by such a transformation only two possibilities can occur. The first one corresponds to the map of the unordered pair ($\pm\mathbf{i}$) into the unordered pair $(0, 2)$. The second one corresponds to the map of the unordered pair $(0, \infty)$ into the unordered pair $(0, 2)$.

Then, one can reduce the transformed equation to the Heun one by a substitution of the form (C.3). One of requirements of the first case of theorem C.1 is the equality of characteristic exponents at points 0 and 2. In terms of characteristic exponents (32) it means that either $\left|\rho_+^{(\mathbf{i})} - \rho_-^{(\mathbf{i})}\right| = \left|\rho_+^{(-\mathbf{i})} - \rho_-^{(-\mathbf{i})}\right|$ or $\left|\rho_+^{(0)} - \rho_-^{(0)}\right| = \left|\rho_+^{(\infty)} - \rho_-^{(\infty)}\right|$. The first possibility cannot occur for a nontrivial γ. Therefore, without losing generality, one can consider the map

$$t = \tau(r) := \frac{2r}{r+\mathbf{i}}, \qquad \tau : (-\mathbf{i}, 0, \mathbf{i}, \infty) \to (\infty, 0, 1, 2).$$

This map transforms equation (30) with potential (31) into the equation

$$f_{tt}(t) + \mathcal{A}(t) f_t(t) - \mathcal{B}(t) f(t) = 0, \qquad |t-1| = 1, \quad \operatorname{Im} t < 0, \tag{34}$$

where

$$\mathcal{A}(t) = \frac{nt^2 - 2nt + 2n - 2}{t(t-1)(t-2)},$$
$$\mathcal{B}(t) = 2\frac{m(ER^2t^2(t-2)^2 + R\gamma\mathbf{i}t(t-2)(t^2-2t+2)) + a(t-2)^4 - bt^2(t-2)^2 + ct^4}{t^2(t-1)^2(t-2)^2}.$$

The substitution

$$f(t) = t^{\rho_+^{(0)}}(t-1)^{\rho_+^{(\mathbf{i})}}(t-2)^{\rho_+^{(\infty)}} w(t)$$

transforms (34) to Heun equation (C.10) with the parameter γ' instead of γ, where

$$\alpha = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \rho_+^{(-\mathbf{i})}, \qquad \beta = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \rho_-^{(-\mathbf{i})}, \qquad d = 2,$$
$$\gamma' = 1 - \rho_-^{(0)} + \rho_+^{(0)}, \qquad \delta = 1 - \rho_-^{(\mathbf{i})} + \rho_+^{(\mathbf{i})}, \qquad \varepsilon = 1 - \rho_-^{(\infty)} + \rho_+^{(\infty)}.$$

Here, $t^{\rho_+^{(0)}}(t-1)^{\rho_+^{(\mathbf{i})}}(t-2)^{\rho_+^{(\infty)}}$ means the function holomorphic on $\mathbb{C}\backslash(-\infty, 2]$ and real for real $t > 2$. Restrictions on asymptotics of the function f near the points $r = 0, \infty$ are equivalent to the boundedness of the function $w(t)$ near the points $t = 0, 2$.

Obviously, the accessory parameter q can be found as

$$q = -2\lim_{t\to 0} t\left(-\mathcal{B}(t) + \left(\frac{\rho_+^{(0)}}{t} + \frac{\rho_+^{(\mathbf{i})}}{t-1} + \frac{\rho_+^{(\infty)}}{t-2}\right)\mathcal{A}(t) + \frac{\rho_+^{(0)}\left(\rho_+^{(0)} - 1\right)}{t^2} + \frac{2\rho_+^{(0)}\rho_+^{(\mathbf{i})}}{t(t-1)}\right.$$
$$\left. + \frac{2\rho_+^{(0)}\rho_+^{(\infty)}}{t(t-2)}\right) = 4\rho_+^{(0)}\rho_+^{(\mathbf{i})} + 2\rho_+^{(0)}\rho_+^{(\infty)} - (n-3)\rho_+^{(0)} + (n-1)\left(2\rho_+^{(\mathbf{i})} + \rho_+^{(\infty)}\right)$$
$$- 4mR\gamma\mathbf{i} + 16a. \tag{35}$$

Theorem C.1 implies that this Heun equation can be transformed into the hypergeometric equation by a rational change of independent variable $t \to z : z = P(t)$, where P is a rational function, iff

$$\gamma' = \varepsilon, \tag{36}$$
$$\alpha\beta - q = 0. \tag{37}$$

Equation (36) is equivalent to

$$a = c. \tag{38}$$

Using the equalities

$$\alpha = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \tfrac{1}{2}\left(n - 1 + \sqrt{(n-1)^2 + 8(mER^2 + \mathbf{i}mR\gamma + a - b + c)}\right),$$
$$\beta = \rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \tfrac{1}{2}\left(n - 1 - \sqrt{(n-1)^2 + 8(mER^2 + \mathbf{i}mR\gamma + a - b + c)}\right),$$

one can rewrite equation (37) as

$$\begin{aligned}\big(\rho_+^{(0)} + \rho_+^{(\mathbf{i})} + \rho_+^{(\infty)} + \tfrac{1}{2}(n-1)\big)^2 - \tfrac{1}{4}((n-1)^2 + 8(mER^2 + \mathbf{i}mR\gamma + a - b + c)) - 4\rho_+^{(0)}\rho_+^{(\mathbf{i})}\\ - 2\rho_+^{(0)}\rho_+^{(\infty)} + (n-3)\rho_+^{(0)} - (n-1)\big(2\rho_+^{(\mathbf{i})} + \rho_+^{(\infty)}\big) + 4mR\gamma\mathbf{i} - 16a\\ = \big(\rho_+^{(0)}\big)^2 + \big(\rho_+^{(\mathbf{i})}\big)^2 + \big(\rho_+^{(\infty)}\big)^2 + 2\rho_+^{(\mathbf{i})}\big(\rho_+^{(\infty)} - \rho_+^{(0)}\big) + (2n-4)\rho_+^{(0)}\\ - (n-1)\rho_+^{(\mathbf{i})} + 2mR\gamma\mathbf{i} - 2mER^2 - 18a + 2b - 2c = 0.\end{aligned} \tag{39}$$

Excluding squares of values $\rho_+^{(0)}, \rho_+^{(\mathbf{i})}, \rho_+^{(\infty)}$ from (39) with the help of obvious equations

$$\big(\rho_+^{(0)}\big)^2 + (n-2)\rho_+^{(0)} - 8a = 0,$$
$$\big(\rho_+^{(\mathbf{i})}\big)^2 - (n-1)\rho_+^{(\mathbf{i})} - 2mR(RE - \gamma\mathbf{i}) - 2(a - b + c) = 0,$$
$$\big(\rho_+^{(\infty)}\big)^2 + (n-2)\rho_+^{(\infty)} - 8c = 0$$

for characteristic exponents, one gets

$$\big(2\rho_+^{(\mathbf{i})} - n + 2\big)\big(\rho_+^{(\infty)} - \rho_+^{(0)}\big) + 8(c - a) = 0.$$

For $a = c$, it holds $\rho_+^{(\infty)} = \rho_+^{(0)}$ and thus equation (37) is a consequence of (38).

From here till the end of the present subsection we suppose that $a = c$. This condition corresponds to cases 1 and 4 of proposition 3, cases 1, 2, 5 and 8 of proposition 4, and cases 1 and 4 of proposition 5.

The fist case of theorem C.1 implies then that the function w w.r.t. a new independent variable

$$z := 1 - (t-1)^2 = t(2-t) \tag{40}$$

satisfies the hypergeometric equation:

$$z(1-z)w''(z) + (\widetilde{\gamma} - (\widetilde{\alpha} + \widetilde{\beta} + 1)z)w'(z) - \widetilde{\alpha}\widetilde{\beta}w(z) = 0. \tag{41}$$

The correspondence between characteristic exponents of the Heun and the hypergeometric equations connected by (40) implies

$$\widetilde{\gamma} = \gamma' = 1 + \sqrt{(n-2)^2 + 32a} \in \mathbb{R}, \qquad \widetilde{\alpha} = \tfrac{1}{2}\alpha = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(n-2)^2 + 32a} + \tfrac{1}{4}(s + \bar{s}) \in \mathbb{R},$$
$$\widetilde{\beta} = \tfrac{1}{2}\beta = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(n-2)^2 + 32a} + \tfrac{1}{4}(-s + \bar{s}) \notin \mathbb{R},$$

where $s = \sqrt{(n-1)^2 + 8(mER^2 + \mathbf{i}mR\gamma + 2a - b)}$.

Since

$$z - 1 = -\left(\frac{r - i}{r + i}\right)^2,$$

the half-line $[0, \infty]$ on the r-plane is mapped into the circumference on the z-plane defined by the equation $|z - 1| = 1$, while the values $r = 0, \infty$ correspond to the point $z = 0$.

The function $w(z)$ is bounded near the point $z = 0$ and $1 - \widetilde{\gamma} = -\sqrt{(n-2)^2 + 32a} \leqslant 0$; therefore it holds that (see (C.5))

$$w(z) = w_+(z) := c_+ F(\widetilde{\alpha}, \widetilde{\beta}; \widetilde{\gamma}; z), \qquad z \in (|z-1| = 1, \operatorname{Im} z > 0), \qquad c_+ = \text{const},$$
$$w(z) = w_-(z) := c_- F(\widetilde{\alpha}, \widetilde{\beta}; \widetilde{\gamma}; z), \qquad z \in (|z-1| = 1, \operatorname{Im} z < 0), \qquad c_- = \text{const}.$$

An equivalent problem for the hypergeometric equation was considered in [14]. However, there was made an assumption equivalent to $c_+ = c_-$ without any proof (formula (3) in [14]). Below we fill this gap.

Functions $w_\pm(z)$ should be analytic continuations of each other through the regular point $z = 2$.[3] Due to formula (C.6) (applicable since $\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta} \notin \mathbb{R}$), it means that functions

$$c_+ \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta})}{\Gamma(\widetilde{\gamma} - \widetilde{\alpha})\Gamma(\widetilde{\gamma} - \widetilde{\beta})} F(\widetilde{\alpha}, \widetilde{\beta}; \widetilde{\alpha} + \widetilde{\beta} - \widetilde{\gamma} + 1; 1 - z), \qquad |z - 1| = 1, \quad \operatorname{Im} z > 0$$

and

$$c_- \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta})}{\Gamma(\widetilde{\gamma} - \widetilde{\alpha})\Gamma(\widetilde{\gamma} - \widetilde{\beta})} F(\widetilde{\alpha}, \widetilde{\beta}; \widetilde{\alpha} + \widetilde{\beta} - \widetilde{\gamma} + 1; 1 - z), \qquad |z - 1| = 1, \quad \operatorname{Im} z < 0$$

are analytic continuations of each other through the point $z = 2$ as well as functions

$$c_+ \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\alpha} + \widetilde{\beta} - \widetilde{\gamma})}{\Gamma(\widetilde{\alpha})\Gamma(\widetilde{\beta})} (1 - z)^{\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta}} F(\widetilde{\gamma} - \widetilde{\alpha}, \widetilde{\gamma} - \widetilde{\beta}; \widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta} + 1; 1 - z),$$
$$|z - 1| = 1, \quad \operatorname{Im} z > 0$$

and

$$c_- \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\alpha} + \widetilde{\beta} - \widetilde{\gamma})}{\Gamma(\widetilde{\alpha})\Gamma(\widetilde{\beta})} (1 - z)^{\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta}} F(\widetilde{\gamma} - \widetilde{\alpha}, \widetilde{\gamma} - \widetilde{\beta}; \widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta} + 1; 1 - z),$$
$$|z - 1| = 1, \quad \operatorname{Im} z < 0.$$

The first requirement is equivalent to the equality

$$(c_+ - c_-) \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta})}{\Gamma(\widetilde{\gamma} - \widetilde{\alpha})\Gamma(\widetilde{\gamma} - \widetilde{\beta})} = 0, \tag{42}$$

while the second one is equivalent to the equality

$$(c_+ - c_- \exp(2\pi \mathbf{i}(\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta}))) \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\alpha} + \widetilde{\beta} - \widetilde{\gamma})}{\Gamma(\widetilde{\alpha})\Gamma(\widetilde{\beta})} = 0. \tag{43}$$

Since $\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta} \notin \mathbb{R}$, linear system (42), (43) has a nontrivial solution c_+, c_- iff either

$$\frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta})}{\Gamma(\widetilde{\gamma} - \widetilde{\alpha})\Gamma(\widetilde{\gamma} - \widetilde{\beta})} = 0 \qquad \text{or} \qquad \frac{\Gamma(\widetilde{\gamma})\Gamma(\widetilde{\alpha} + \widetilde{\beta} - \widetilde{\gamma})}{\Gamma(\widetilde{\alpha})\Gamma(\widetilde{\beta})} = 0.$$

Taking into account $\widetilde{\gamma} - \widetilde{\beta} \notin \mathbb{R}$, $\widetilde{\beta} \notin \mathbb{R}$, one gets $\widetilde{\gamma} - \widetilde{\alpha} = -k + 1$ or $\widetilde{\alpha} = -k + 1$, $k \in \mathbb{N}$.

Not losing generality suppose that $\operatorname{Re} s < 0$. Then, the first equality is impossible and the second one yields

$$s = 1 - 2k - \sqrt{(n-2)^2 + 32a} + \frac{4\mathbf{i}mR\gamma}{1 - 2k - \sqrt{(n-2)^2 + 32a}},$$

since $\operatorname{Im} s^2 = 8\mathbf{i}mR\gamma$. From the definition of s, one gets therefore the following formula for energy levels:

$$E_k = \frac{1}{mR^2}\left(\frac{1}{2}(k^2 - k + 1) - \frac{n}{4} + 2a + b + \frac{2k-1}{4}\sqrt{(n-2)^2 + 32a}\right)$$
$$- \frac{2m\gamma^2}{(\sqrt{(n-2)^2 + 32a} + 2k - 1)^2}, \qquad k \in \mathbb{N}.$$

These energy levels are degenerated and their multiplicities coincide with multiplicities of eigenvalues in propositions 3–5.

[3] Recall that the function $F(\alpha', \beta'; \gamma'; z)$ is holomorphic in $\mathbb{C}\backslash[1, +\infty)$.

Taking into account all transformations used while reducing equation (30) to the hypergeometric one, we get the following expression for radial eigenfunctions (up to an arbitrary constant nonzero factor):

$$f_k(r) = \frac{r^{\rho_+^{(0)}}(r-\mathbf{i})^{\rho_+^{(\mathbf{i})}}}{(r+\mathbf{i})^{\rho_+^{(0)}+\rho_+^{(\mathbf{i})}+\rho_+^{(\infty)}}}\sum_{j=0}^{k-1}\frac{(-1)^j}{j!(k-j-1)!}\frac{(\widetilde{\beta})_j}{(\widetilde{\gamma})_j}\frac{(4r\mathbf{i})^j}{(r+\mathbf{i})^{2j}},$$

where $\rho_+^{(0)}$, $\rho_+^{(\mathbf{i})}$, $\rho_+^{(\infty)}$, $\widetilde{\beta}$ and $\widetilde{\gamma}$ are given by above formulae for $E = E_k$.

6.2. *Oscillator potential*

The oscillator potential for the sphere $\mathbf{S}^n$ has the form

$$V_o(r) = \frac{1}{2}R^2\omega^2\tan^2\frac{\rho}{R} = \frac{2R^2\omega^2r^2}{(1-r^2)^2}, \qquad \omega \in \mathbb{R}_+.$$

It has a positive singularity along the sphere equator and looks like an infinite potential well. Therefore, from the physical point of view it is natural to consider wavefunctions defined on M' and vanishing as $r \to 1$.

From the mathematical point of view theorem B.1 is not applicable since

$$V_o \notin \mathcal{L}^1_{\mathrm{loc}}(\mathbf{S}^n \times \mathbf{S}^n, \chi \times \chi).$$

However, since $V_o \geqslant 0$ one can use the Friedrichs extension $(H_{V_o})_F$ of a Hamiltonian with the domain given by theorem B.2, where $M' \subset \mathbf{S}^n \times \mathbf{S}^n$ is defined by the inequality $r = \tan(\rho/(2R)) < 1$.

Equation (30) for $V = V_o$ is a Fuchsian one with six singular points $0, \pm 1, \pm\mathbf{i}, \infty$ and corresponding characteristic exponents:

$$\rho_\pm^{(0)} = \tfrac{1}{2}\left(2-n\pm\sqrt{(n-2)^2+32a}\right), \qquad \rho_\pm^{(\infty)} = \tfrac{1}{2}\left(2-n\pm\sqrt{(n-2)^2+32c}\right),$$
$$\rho_\pm^{(\mathbf{i})} = \rho_\pm^{(-\mathbf{i})} = \tfrac{1}{2}(n-1)\pm\tfrac{1}{2}\sqrt{(n-1)^2+8mER^2+4mR^4\omega^2+8(a-b+c)},$$
$$\rho_\pm^{(1)} = \rho_\pm^{(-1)} = \tfrac{1}{2}\left(1\pm\sqrt{1+4R^4m\omega^2}\right).$$

Similarly to the previous section the function $f(r)$, $r \in (0,1)$ should be $\sim r^{\rho_+^{(0)}}$ as $r \to +0$. On the other hand, the inclusion

$$f(r)\psi_D \in W^{1,2}(M', \chi \times \chi)$$

implies the convergence of the integral

$$\int_{M'} g_2(\nabla\overline{(f\psi_D)}, \nabla(f\psi_D))\,\mathrm{d}\chi \times \mathrm{d}\chi, \tag{44}$$

where g_2 is defined in (5) and ∇ means the gradient operator.

The convergence of (44) is equivalent to the convergence of its 'radial part'

$$\int_0^1 |f'(r)|^2\frac{r^{n-1}\,\mathrm{d}r}{(1+r^2)^{n-4}}.$$

Therefore, if $f \sim r^{\rho^{(1)}}$ as $r \to 1-0$, then $\rho^{(1)} > 1/2$ and thus $\rho^{(1)} = r^{\rho_+^{(1)}}$.

Conversely, it can be easily verified that if f is a solution of (30) with asymptotics $f(r) \sim r^{\rho_+^{(0)}}$ as $r \to 0$ and $f(r) \sim r^{\rho_+^{(1)}}$ as $r \to 1-0$ for $V = V_o$, then $f(r)\psi_D \in \mathrm{Dom}\big((H_{V_o})_F\big)$.

Fortunately, one can glue points $r = \pm 1$ together (as well as points $r = \pm\mathbf{i}$) by the change of the independent variable $r \to \zeta$, $\zeta = r^2$, which transforms the differential equation under consideration into the following Fuchsian differential equation with four singular points:

$$f_{\zeta\zeta} + \frac{n+(4-n)\zeta}{2\zeta(\zeta+1)} f_\zeta + \frac{2}{\zeta(\zeta+1)^2}\left(mR^2\left(E - \frac{2R^2\omega^2\zeta}{(\zeta-1)^2}\right) - \frac{a}{\zeta} - b - c\zeta\right) f = 0,$$
$$0 < \zeta < 1. \tag{45}$$

The singular points $-1, 0, 1, \infty$ of this equation form a harmonic quadruple and correspond, respectively, to the following characteristic exponents

$$\rho_\pm^{(\mathbf{i})}, \qquad \tfrac{1}{2}\rho_\pm^{(0)}, \qquad \rho_\pm^{(1)}, \qquad \tfrac{1}{2}\rho_\pm^{(\infty)}.$$

The same arguments as for the Coulomb problem leads to the conclusion that the only possibility to transform equation (45) to the hypergeometric one via transformations (C.2), (C.3) and then using theorem C.1 corresponds to the map of the unordered pair $(0, \infty)$ into the unordered pair $(0, 2)$ by a Möbius transformation.

Without losing generality, one can consider the substitution

$$t = \tau(\zeta) = \frac{2\zeta}{\zeta+1}, \qquad \tau\colon (-1, 0, 1, \infty) \to (\infty, 0, 1, 2). \tag{46}$$

The interval under consideration for the variable t is again $(0, 1)$. Substitution (46) transforms equation (45) into equation (34) with

$$\mathcal{A}(t) = \frac{n(t-1)}{t(t-2)}, \quad \mathcal{B}(t) = \frac{2}{t(t-2)}\left(mR^2\left(E + \frac{R^2\omega^2 t(t-2)}{2(t-1)^2}\right) - \frac{2a}{t} + a - b + \frac{ct}{t-2}\right).$$

Define a function $w(t)$ by

$$w(t) = t^{-\frac{1}{2}\rho_+^{(0)}}(t-1)^{-\rho_+^{(1)}}(t-2)^{-\frac{1}{2}\rho_+^{(\infty)}} f(t).$$

It satisfies Heun equation (C.10), where

$$\alpha = \tfrac{1}{2}\rho_+^{(0)} + \rho_+^{(1)} + \tfrac{1}{2}\rho_+^{(\infty)} + \rho_+^{(\mathbf{i})}, \qquad \beta = \tfrac{1}{2}\rho_+^{(0)} + \rho_+^{(1)} + \tfrac{1}{2}\rho_+^{(\infty)} + \rho_-^{(\mathbf{i})}, \qquad d = 2,$$
$$\gamma = 1 + \tfrac{1}{2}\big(\rho_+^{(0)} - \rho_-^{(0)}\big), \qquad \delta = 1 + \rho_+^{(1)} - \rho_-^{(1)}, \qquad \varepsilon = 1 + \tfrac{1}{2}\big(\rho_+^{(\infty)} - \rho_-^{(\infty)}\big).$$

Here, $t^{-\frac{1}{2}\rho_+^{(0)}}(t-1)^{-\rho_+^{(1)}}(t-2)^{-\frac{1}{2}\rho_+^{(\infty)}}$ means the function holomorphic on the domain $\mathbb{C}\backslash((-\infty, 0] \cup [1, +\infty))$ and real for real $t \in (0, 1)$. Restrictions on asymptotics of the function $f(r)$ near the points $r = 0, 1$ are equivalent to the boundedness of the function $w(t)$ near the points $t = 0, 1$.

Calculation, similar to (35), yields the following value of accessory parameter q for (C.10):

$$q = -2mR^2E + 2b + n\left(\rho_+^{(1)} + \frac{1}{4}\rho_+^{(\infty)}\right) + 2\rho_+^{(0)}\rho_+^{(1)} + \frac{1}{2}\rho_+^{(0)}\rho_+^{(\infty)} + \frac{n}{4}\rho_+^{(0)}.$$

Condition (36) of theorem C.1 is again equivalent to (38). Condition (37) of the same theorem can be written as

$$\alpha\beta - q = \rho_+^{(1)}\big(\rho_+^{(\infty)} - \rho_+^{(0)}\big) = 0,$$

which is again a consequence of (36).

Suppose that condition (36) is valid. Thus, we are in the situation of the first case of theorem C.1 and changing the independent variable t by a new one z according to (40), one gets hypergeometric equation (41) with

$$\widetilde{\alpha} = \tfrac{1}{2}\alpha = \tfrac{1}{4}\big(2 + \sqrt{(n-2)^2 + 32a} + \sqrt{1 + 4R^4 m\omega^2} + s\big),$$
$$\widetilde{\beta} = \tfrac{1}{2}\beta = \tfrac{1}{4}\big(2 + \sqrt{(n-2)^2 + 32a} + \sqrt{1 + 4R^4 m\omega^2} - s\big),$$
$$\widetilde{\gamma} = \gamma = 1 + \tfrac{1}{2}\sqrt{(n-2)^2 + 32a},$$

where $s = \sqrt{(n-1)^2 + 8mER^2 + 4mR^4\omega^2 + 16a - 8b}$. The interval $(0, 1) \ni t$ corresponds to the interval $(0, 1) \ni z$, therefore the requirement on asymptotic of the function $f(t)$ near the point $t = 0$ implies

$$w(z) = F(\widetilde{\alpha}, \widetilde{\beta}; \widetilde{\gamma}; z).$$

Also due to

$$\widetilde{\gamma} - \widetilde{\alpha} - \widetilde{\beta} = -\tfrac{1}{2}\sqrt{1 + 4R^4 m\omega^2} < 0, \qquad \operatorname{Re}\widetilde{\alpha} > 0$$

and (C.9), the requirement on asymptotic of the function $f(t)$ near the point $t = 1$ implies

$$\widetilde{\beta} = -k, \qquad k = 0, 1, 2, \ldots.$$

This leads to energy levels

$$E_k = \frac{1}{8mR^2}((4k + 2 + \sqrt{(n-2)^2 + 32a})^2 - (n-1)^2 - 16a + 8b + 1)$$
$$+ \frac{\omega}{2\sqrt{m}}(4k + 2 + \sqrt{(n-2)^2 + 32a})\sqrt{1 + \frac{1}{4R^4m^2}}, \qquad k = 0, 1, 2, \ldots.$$

Again multiplicities of these energy levels coincide with multiplicities of eigenvalues in propositions 3–5.

The expression for radial eigenfunctions (up to an arbitrary constant nonzero factor) is

$$f_k(r) = \frac{r^{\rho_+^{(0)}}(r^2-1)^{\rho_+^{(1)}}}{(r^2+1)^{\frac{1}{2}\rho_+^{(0)} + \rho_+^{(1)} + \frac{1}{2}\rho_+^{(\infty)}}} \sum_{j=0}^{k} \frac{(-1)^j}{j!(k-j)!} \frac{(\widetilde{\alpha})_j}{(\widetilde{\gamma})_j} \frac{4^j r^{2j}}{(r^2+1)^{2j}},$$

where $\rho_+^{(0)}, \rho_+^{(1)}, \rho_+^{(\infty)}, \widetilde{\alpha}$ and $\widetilde{\gamma}$ are given by above formulae for $E = E_k$.

7. Conclusion

The possibility to find in an explicit way some (but not all) eigenvalues for a Schrödinger operator characterizes so-called quasi-exactly solvable models [43–45]. In the present paper, we have shown that the two-body problem on spheres $\mathbf{S}^n$ with Coulomb and oscillator potentials is quasi-exactly solvable for any n. A possible generalization for other compact two-point homogeneous spaces is an open problem.

The quasi-exactly solvability here is an attribute not of a radial differential equation (30), but of the whole problem. It stems from the two causes. The first cause follows from the fact that we restrict our consideration on the subspace of $\mathcal{L}(G, K, \mu)$ (see (12)) consisting of common eigenfunctions for operators D_0^2, D_1, D_2 and optionally D_3. For every such eigenfunction one gets a separate radial differential equation (30). For the Coulomb and oscillator potentials this equation can be reduced to the Heun one, but the further reduction to the hypergeometric equation using Maier's scheme is possible only for some eigenfunctions (just for those that satisfies equation (38)) and this is the second cause.

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Appendix A. Orthogonal complex Lie algebras and their representations

A.1. Lie algebra $\mathfrak{B}_k$

Here is a brief description of the simple complex Lie algebra $\mathfrak{B}_k \cong \mathfrak{so}(2k+1, \mathbb{C})$ (see [46, 47] and [48] for details).

Denote

$$S_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 1 & 0 & \dots & 0 & 0 & 0 \end{pmatrix} \in GL(i), \qquad i \in \mathbb{N}.$$

Consider the Lie algebra $\mathfrak{B}_k \cong \mathfrak{so}(2k+1, \mathbb{C})$ as

$$\mathfrak{B}_k = (A \in \mathfrak{gl}(2k+1, \mathbb{C}) | A^T S_{2k+1} + S_{2k+1} A = 0). \tag{A.1}$$

Following [36], we shall enumerate the rows and columns of $A \in \mathfrak{B}_k$ by the indices $-k, \dots, -1, 0, 1, \dots, k$. The convenience of such notation is due to the fact that subalgebras $\mathfrak{B}_i \subset \mathfrak{B}_k$, $i < k$, correspond to indices of rows and columns from $-i$ to i.

It can be easily shown that a matrix

$$A = \sum_{i,j} a_{ij} E_{ij} \in \mathfrak{gl}(2k+1, \mathbb{C})$$

belongs to $\mathfrak{B}_k$ iff $a_{ij} + a_{-j,-i} = 0$, which means that A is skew-symmetric w.r.t. its secondary diagonal.

Let $F_{ij} = E_{ij} - E_{-j,-i}$. It is easily seen that

$$[F_{ij}, F_{pq}] = \delta_{jp} F_{iq} - \delta_{iq} F_{pj} + \delta_{-pi} F_{-qj} + \delta_{-jq} F_{p,-i}.$$

The algebra $\mathfrak{B}_k$ is spanned by elements F_{ij} with $i > -j$. Evidently, $F_{i,-i} = 0$ and $F_{-j,-i} = -F_{ij}$.

Elements F_{ii}, $i = 1, \dots, k$ form a base of the Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{B}_k$, which consists of elements of the form

$$X = \operatorname{diag}(-x_k, -x_{k-1}, \dots, -x_1, 0, x_1, \dots, x_{k-1}, x_k).$$

Let $\varepsilon_i \in \mathfrak{h}_k^*$ such that $\varepsilon_i(X) = x_i$, i.e. ε_i is a base in $\mathfrak{h}_k^*$ dual to $F_{i,i}$, $i = 1, \dots, k$. Define a symmetric nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{B}_k$ as

$$\langle A, B \rangle = \tfrac{1}{2} \operatorname{tr} AB, \tag{A.2}$$

which is proportional to the Killing form. Clearly,

$$\langle F_{ij}, F_{qp} \rangle = \delta_{ip} \delta_{jq}, \qquad i > -j, \qquad q > -p.$$

In particular, F_{ii}, $i = 1, \dots, k$, is an orthogonal base in $\mathfrak{h}_k$.

The form $\langle \cdot, \cdot \rangle|_{\mathfrak{h}_k}$ generates the isomorphism $\varkappa : \mathfrak{h}_k \to \mathfrak{h}_k^*$ by the formula $\varkappa(X) = \langle X, \cdot \rangle$. Specifically, $\varkappa(F_{i,i}) = \varepsilon_i$ and ε_i, $i = 1, \dots, k$ is an orthonormal base in $\mathfrak{h}_k^*$ w.r.t. the form

$$\langle f_1, f_2 \rangle^* := \langle \varkappa^{-1}(f_1), \varkappa^{-1}(f_2) \rangle, \qquad f_1, f_2 \in \mathfrak{h}_k^*.$$

Using this notation one can describe the standard form of the root system for $\mathfrak{B}_k$ in the following way. Let

$$\Phi_{\mathfrak{B}_k} := (\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i \neq j, i, j = 1, \dots, k)$$

be a root system in $\mathfrak{B}_k$,

$$\Phi^+_{\mathfrak{B}_k} := (\varepsilon_i, \varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j \mid i > j, i, j = 1, \dots, k)$$

be a system of positive roots and

$$\Delta_{\mathfrak{B}_k} := (\alpha_1 = \varepsilon_1, \alpha_i = \varepsilon_i - \varepsilon_{i-1} \mid i = 2, \dots, k)$$

be a system of simple roots, corresponding to the inverse lexicographic order. A subalgebra $\mathfrak{B}_i \subset \mathfrak{B}_k$, $i < k$, corresponds to root systems $\Phi_{\mathfrak{B}_i}$, $\Phi^+_{\mathfrak{B}_i}$ and $\Delta_{\mathfrak{B}_i}$.

Let L_α be a root subspace in $\mathfrak{B}_k$, corresponding to a root $\alpha \in \Phi_{\mathfrak{B}_k}$. Then,

$$\begin{aligned} &L_{-\varepsilon_i} = \mathrm{span}(F_{0i}), && L_{\varepsilon_i} = \mathrm{span}(F_{i0}), && L_{\varepsilon_i-\varepsilon_j} = \mathrm{span}(F_{ij}), \\ &L_{\varepsilon_i+\varepsilon_j} = \mathrm{span}(F_{i,-j}), && L_{-\varepsilon_i-\varepsilon_j} = \mathrm{span}(F_{-ij}), && i, j = 1, \dots, k. \end{aligned} \tag{A.3}$$

Fundamental weights for $\mathfrak{B}_k$ are

$$\lambda_1 = \frac{1}{2}\sum_{j=1}^{k} \varepsilon_j, \qquad \lambda_i = \sum_{j=i}^{k} \varepsilon_j, \qquad i = 2, \dots, k.$$

Let

$$\lambda = \sum_{j=1}^{k} \lambda^j \lambda_j, \qquad \lambda^j \in \mathbb{Z}_+ := (0) \cup \mathbb{N}$$

be a dominant weight and $V(\lambda)$ be an irreducible finite-dimensional $\mathfrak{B}_k$-module with the highest weight λ. All finite-dimensional irreducible representations of $\mathfrak{B}_k$ are of this form, modules $V(\lambda)$ with different λ are not isomorphic to each other and $V(\lambda)$ corresponds to a (single-valued) representation of the group $SO(2k + 1)$ iff λ_1 is even. The dominant weight λ can be written in the form

$$\lambda = \sum_{i=1}^{k} m_i \varepsilon_i, \qquad m_k \geqslant m_{k-1} \geqslant \cdots \geqslant m_1 \geqslant 0, \tag{A.4}$$

where either all $m_i \in \mathbb{Z}_+$ or all $m_i \in \mathbb{Z}_+ + \frac{1}{2}$. Even values of λ_1 corresponds to $m_i \in \mathbb{Z}_+$. Let δ be the sum of fundamental weights. Then it holds

$$\delta = \sum_{i=1}^{k} \lambda_i = \frac{1}{2} \sum_{\alpha \in \Phi^+_{\mathfrak{B}_k}}^{k} \alpha = \sum_{i=1}^{k} \left(i - \frac{1}{2}\right) \varepsilon_i. \tag{A.5}$$

The universal Casimir operator $C \in U(\mathfrak{B}_k)$ is

$$C = \sum_{i=1}^{k} \left(F_{ii}^2 + \{F_{i0}, F_{0i}\}\right) + \sum_{i>j>0} (\{F_{ij}, F_{ji}\} + \{F_{i,-j}, F_{-ji}\}). \tag{A.6}$$

The following formulae are valid for any semisimple Lie algebra:

$$C|_{V(\lambda)} = (\langle \delta + \lambda, \delta + \lambda \rangle - \langle \delta, \delta \rangle)\, \mathrm{id}, \tag{A.7}$$

$$\dim V(\lambda) = \prod_{\alpha \succ 0} \langle \lambda + \delta, \alpha \rangle \Big/ \prod_{\alpha \succ 0} \langle \delta, \alpha \rangle, \tag{A.8}$$

where $\alpha \succ 0$ means a positive root.

For any semisimple Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$, the module $V(\lambda)$ can be decomposed into the finite direct sum of weight subspaces

$$V(\lambda) = \bigoplus_{\mu} V_\mu(\lambda), \qquad \mu \in \mathfrak{h}^*,$$

where $\forall v \in V_\mu(\lambda)$, $\forall h \in \mathfrak{h}$, it holds $h(v) = \mu(h)v$ and the sum is over weights of the form

$$\lambda - \sum_{\alpha \succ 0} i_\alpha \alpha, \qquad i_\alpha \in \mathbb{Z}_+.$$

Besides, for any root α of $\mathfrak{g}$ one has

$$\xi_\alpha : V_\mu(\lambda) \to V_{\mu+\alpha}(\lambda), \qquad \xi_\alpha \in L_\alpha. \tag{A.9}$$

A.2. Lie algebra $\mathfrak{D}_k$

The Lie algebra $\mathfrak{D}_k$ is the subalgebra of $\mathfrak{B}_k$, consisting of matrices whose column and rows with the index 0 vanish. We shall discard these null row and column and shall enumerate other rows and columns of $A \in \mathfrak{D}_k$ by the indices $-k, \dots, -1, 1, \dots, k$ as before. The Cartan subalgebra $\mathfrak{h}_k \subset \mathfrak{D}_k$ is the same as in the $\mathfrak{B}_k$-case. Describe the $\mathfrak{D}_k$-case briefly, emphasizing differences from the $\mathfrak{B}_k$-case.

Now one has

$$\begin{aligned}
\Phi_{\mathfrak{D}_k} &:= (\pm\varepsilon_i \pm \varepsilon_j | i \neq j, i, j = 1, \dots, k),\\
\Phi^+_{\mathfrak{D}_k} &:= (\varepsilon_i + \varepsilon_j, \varepsilon_i - \varepsilon_j | i > j, i, j = 1, \dots, k),\\
\Delta_{\mathfrak{D}_k} &:= (\alpha_1 = \varepsilon_1 + \varepsilon_2, \alpha_i = \varepsilon_i - \varepsilon_{i-1} | i = 2, \dots, k).
\end{aligned}$$

The root subspaces $L_{\pm\varepsilon_i \pm\varepsilon_j}$ are the same as in $\mathfrak{B}_k$-case.

Fundamental weights are

$$\lambda_1 = \frac{1}{2}\sum_{j=1}^{k}\varepsilon_j, \qquad \lambda_2 = -\frac{1}{2}\varepsilon_1 + \frac{1}{2}\sum_{j=2}^{k}\varepsilon_j, \qquad \lambda_i = \sum_{j=i}^{k}\varepsilon_j, \qquad i = 3, \dots, k.$$

The sum of fundamental weights is

$$\delta = \sum_{i=1}^{k}\lambda_i = \frac{1}{2}\sum_{\alpha \in \Phi^+_{\mathfrak{D}_k}}\alpha = \sum_{i=2}^{k}(i-1)\varepsilon_i.$$

A dominant weight

$$\lambda = \sum_{j=1}^{k}\lambda^j\lambda_j, \lambda^j \in \mathbb{Z}_+ := (0) \cup \mathbb{N}$$

now has the form

$$\lambda = \sum_{i=1}^{k} m_i\varepsilon_i, \qquad m_k \geqslant m_{k-1} \geqslant \dots \geqslant m_2 \geqslant |m_1|, \tag{A.10}$$

where either $m_1 \in \mathbb{Z}, m_i \in \mathbb{Z}_+, i \geqslant 2$ or $m_1 \in \mathbb{Z} + \frac{1}{2}, m_i \in \mathbb{Z}_+ + \frac{1}{2}, i \geqslant 2$. Again $\mathfrak{D}_k$-modules with integer $m_j, j = 1, \dots, k$, correspond to (single-valued) representations of the group $SO(2k)$.

The universal Casimir operator $C \in U(\mathfrak{D}_k)$ is

$$C = \sum_{i=1}^{k} F_{ii}^2 + \sum_{i>j>0}(\{F_{ij}, F_{ji}\} + \{F_{i,-j}, F_{-ji}\}). \tag{A.11}$$

A.3. Restrictions of $\mathfrak{B}_k$*- and* $\mathfrak{D}_k$*-representations*

The following results were found in [49] (see also [50]).

Let $V_{\mathfrak{B}_k}(\lambda)$ be a simple $\mathfrak{B}_k$-module with a highest weight (A.4) and $V_{\mathfrak{D}_k}(\lambda)$ be a simple $\mathfrak{D}_k$-module with a highest weight

$$\lambda' = \sum_{i=1}^{k} m_i'\varepsilon_i, \qquad m_k' \geqslant m_{k-1}' \geqslant \dots \geqslant m_2' \geqslant |m_1'|.$$

Proposition A.1. *The restriction $V_{\mathfrak{B}_k}(\lambda)|_{\mathfrak{D}_k}$ of the irreducible $\mathfrak{B}_k$-representation onto any subalgebra $\mathfrak{D}_k \subset \mathfrak{B}_k$ expands as follows:*

$$V_{\mathfrak{B}_k}(\lambda)|_{\mathfrak{D}_k} = \bigoplus_{\lambda'} V_{\mathfrak{D}_k}(\lambda'),$$

where the summation is over all λ' such that

$$m_k \geqslant m'_k \geqslant m_{k-1} \geqslant \cdots \geqslant m'_2 \geqslant m_1 \geqslant m'_1 \geqslant -m_1$$

and all m'_j are integer or half integer simultaneously with m_j.

Let $V_{\mathfrak{B}_{k-1}}(\lambda')$ be a simple $\mathfrak{B}_{k-1}$-module with a highest weight

$$\lambda' = \sum_{i=1}^{k-1} m'_i \varepsilon_i, \qquad m'_{k-1} \geqslant m'_{k-2} \geqslant \cdots \geqslant m'_2 \geqslant m'_1 \geqslant 0.$$

Proposition A.2. *The restriction $V_{\mathfrak{D}_k}(\lambda)|_{\mathfrak{B}_{k-1}}$ of the irreducible $\mathfrak{D}_k$-representation onto any subalgebra $\mathfrak{B}_{k-1} \subset \mathfrak{D}_k$ expands as follows:*

$$V_{\mathfrak{D}_k}(\lambda)|_{\mathfrak{B}_{k-1}} = \bigoplus_{\lambda'} V_{\mathfrak{B}_{k-1}}(\lambda'),$$

where the summation is over all λ' such that

$$m_k \geqslant m'_{k-1} \geqslant m_{k-1} \geqslant \cdots \geqslant m_2 \geqslant m'_1 \geqslant |m_1|$$

and all m'_j are integer or half integer simultaneously with m_j.

Appendix B. Self-adjointness of Schrödinger operators on Riemannian spaces

Here, we shall formulate two results concerning the self-adjointness of Schrödinger operators on Riemannian spaces, which is used in section 6.

The first theorem is a result from [51], restricted onto the scalar case.

Theorem B.1. *Let M be a Riemannian manifold of a bounded geometry,* $\dim M = \ell$, *and μ be the measure on M generated by its metric. Also suppose that the potential V can be represented in the form $V = V_1 + V_2$, where real-valued functions V_1, V_2 are as follows: $0 \leqslant V_1 \in \mathcal{L}^1_{\mathrm{loc}}(M,\mu)$, $0 \geqslant V_2 \in \mathcal{L}^p(M,\mu)$ for $p = \ell/2$ if $\ell \geqslant 3$, for $p > 1$ if $\ell = 2$, and for $p = 1$ if $\ell = 1$.*

Then the operator $H_V = -\triangle + V$ is self-adjoint with the domain

$$\mathrm{Dom}(H_V) = \left(u \in W^{1,2}(M,\mu) \middle| \int_M V_1 |u|^2 \,\mathrm{d}\mu < +\infty, H_V u \in \mathcal{L}^2(M,\mu)\right), \tag{B.1}$$

where $H_V u$ is understood in the sense of distributions. Here, $W^{1,2}(M,\mu)$ is the Sobolev space, consisting of functions on M that are in $\mathcal{L}^2(M,\mu)$ with their first derivatives.

Also $Vu \in \mathcal{L}^1_{\mathrm{loc}}(M,\mu)$ for $u \in$ $\mathrm{Dom}(H_V)$.

The definition of a Riemannian manifold of a bounded geometry can be found in [52]. Note that compact and homogeneous Riemannian manifold is always of a bounded geometry.

If the potential V is not in $\mathcal{L}^1_{\mathrm{loc}}(M^n,\mu)$ then theorem B.1 is not applicable. If instead V is bounded from below, one can try to restrict the Schrödinger operator onto some submanifold M' of M^ℓ such that $V|_{M'} \in \mathcal{L}^1_{\mathrm{loc}}(M',\mu)$ and construct the Friedrichs self-adjoint extension [53] of $-\triangle + V$ from the initial domain $C_c^\infty(M')$. This procedure is physically motivated for instance in the case when $V \to +\infty$ near the boundary of M' and therefore wave functions should vanish near this boundary.

Let us turn to the accurate mathematical description. Let M' be an open connected submanifold of a Riemannian space M^ℓ of dimension ℓ with a metric g and an induced measure μ. We do not suppose that M' is complete w.r.t. the Riemannian structure induced by the Riemannian structure on M^ℓ. Let $V \geqslant C \in \mathbb{R}$ be a real-valued function from $\mathcal{L}^1_{\mathrm{loc}}(M', \mu)$ and $H' = -\triangle + V$ be a Schrödinger operator with the domain $C_c^\infty(M')$, consisting of all infinitely smooth complex-valued functions in M' with compact supports. Not losing generality we suppose that $C = 1$. Let $H_F \geqslant \mathrm{id}$ be the abstract Friedrichs extension of H' [53]. We need a precise description of $\mathrm{Dom}(H_F)$.

The operator H' generates sesquilinear nonnegative form $q_{H'}$ by the equality

$$q_{H'}(\varphi, \psi) = \int_{M'} (\overline{H'\varphi})\psi \,\mathrm{d}\mu$$

with the domain $C_c^\infty(M')$. Evidently, its closure is

$$q_{H_F}(\varphi, \psi) = \int_{M'} (g(\nabla\bar{\varphi}, \nabla\psi) + V\bar{\varphi}\psi)\,\mathrm{d}\mu \tag{B.2}$$

with $\mathrm{Dom}(q_{H_F}) \subset \mathcal{L}^2(M', \mu)$ being a closure of $C_c^\infty(M')$ w.r.t. the inner product (B.2), where ∇ is the gradient operator given in local coordinates by the equality

$$\nabla\psi = g^{jk}\frac{\partial\psi}{\partial x^k}\frac{\partial}{\partial x^j}.$$

The operator H_F is defined by the identity

$$\int_{M'} (g(\nabla\bar{\varphi}, \nabla\psi) + V\bar{\varphi}\psi)\,\mathrm{d}\mu = \int_{M'} \bar{\varphi} H_F\psi \,\mathrm{d}\mu, \qquad \forall\varphi \in \mathrm{Dom}\big(q_{H_F}\big), \qquad \psi \in \mathrm{Dom}(H_F).$$

Thus,

$$H_F\psi = (-\triangle\,\psi + V\psi)_{\mathrm{dist}}, \qquad \psi \in \mathrm{Dom}(H_F). \tag{B.3}$$

Theorem B.2. *The domain of the operator H_F is*

$$\big(\psi \in W^{1,2}(M', \mu)\big| V\psi \in \mathcal{L}^1_{\mathrm{loc}}(M', \mu); (-\triangle\,\psi + V\psi)_{\mathrm{dist}} \in \mathcal{L}^2(M', \mu)\big)$$

and H_F acts by formula (B.3).

The proof of this theorem repeats *mutatis mutandis* the proof of theorem X.27 from [53] using the generalization of the *Kato inequality* for Riemannian spaces [54].

Appendix C. Some Fuchsian differential equations

For convenience of references we collected here basic facts concerning some Fuchsian differential equations of the second order: the Riemannian equation and the reducibility of the Heun equation to the hypergeometric one. For details see [55–60].

The linear differential equation

$$w''(z) + p_1(z)w'(z) + p_2(z)w(z) = 0 \tag{C.1}$$

on the Riemannian sphere $\overline{\mathbb{C}} = \mathbf{P}^1(\mathbb{C})$ with meromorphic coefficients $p_i(z), i = 1, 2$, is *Fuchsian* [55] iff

$$p_i(z) = \frac{q_i(z)}{\prod_{k=1}^m (z - z_k)^i}$$

for some singular points $z_1, \ldots, z_m \in \mathbb{C}$ and polynomials $q_i(z)$ of degrees $\leqslant i(m-1)$, $q_i(z_k) \neq 0$. One can find *characteristic exponents* $\rho^{(z_k)}$ of (C.1) at the point z_k by the substitution

$w(z) = (z - z_k)^{\rho^{(z_k)}}$ into (C.1) and keeping only leading terms as $z \to z_k$. This procedure gives a quadratic equation for $\rho^{(z_k)}$. Denote by $\rho_i^{(z_k)}, i = 1, 2$ its solutions for all points $z_k, k = 1, \ldots, m$. The substitution $w(z) = z^{-\rho^{(\infty)}}$ similarly gives characteristic exponents $\rho_1^{(\infty)}, \ldots, \rho_n^{(\infty)}$ in the infinity.

An information on singular points and corresponding characteristic exponents of equation (C.1) can be encoded in the *Riemann P-symbol* $P\{\mathcal{A}; z\}$, where the first row of a matrix $\mathcal{A}$ consists of singular points and other rows of $\mathcal{A}$ consist of corresponding characteristic exponents.

Equation (C.1) with three singular points is called the *Riemannian equation*. Coefficients of the Riemann equation are completely defined by its characteristic exponents.

There are two types of variable change, transforming any Fuchsian equation into another Fuchsian equation. The first one is a *linear-fractional (Möbius) transformation* of the independent variable:

$$z \to t, \qquad z = \frac{\alpha t + \beta}{\gamma t + \delta}, \qquad \alpha\delta - \beta\gamma \neq 0. \tag{C.2}$$

By such transformation one can move three singular points into three arbitrary points of $\overline{\mathbb{C}}$ with the same characteristic exponents.

The second one is a linear transformation of the dependent variable

$$w(z) \to w_1(z) = \left(\frac{z - z_1}{z - z_2}\right)^q w(z), \tag{C.3}$$

which conserves singular points, but changes the characteristic exponents

$$\rho_i^{(z_1)} \to \rho_i^{(z_1)} + q, \qquad \rho_i^{(z_2)} \to \rho_i^{(z_2)} - q, \qquad i = 1, 2.$$

Using these transformation for the Riemannian equation one can move three singular points into the triple $(0, 1, \infty)$ such that $\rho_1^{(0)} = \rho_1^{(1)} = 0$. If one denote $\rho_1^{(\infty)} = \alpha$, $\rho_2^{(\infty)} = \beta$ and $\rho_2^{(0)} = 1 - \gamma$, then the Fuchs identity for this equation gives $\rho_2^{(1)} = \gamma - \alpha - \beta$ that corresponds to the *hypergeometric* or *Gauss equation*:

$$z(1 - z)w''(z) + (\gamma - (\alpha + \beta + 1)z)w'(z) - \alpha\beta w(z) = 0. \tag{C.4}$$

The P-symbol of equation (C.4) is

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & ; z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{matrix}\right\}.$$

Many quantum mechanical problems for constant curvature spaces can be reduced to this equation, while their Euclidean counterparts lead to its limiting cases, obtained from (C.4) by confluence of singular points (such equations are not Fuchsian).

We shall consider only solutions of (C.4) in the case $\gamma \neq -m, m \in \mathbb{N}$. Solutions of (C.1), corresponding to different characteristic exponents near some singular point, are called *canonical solutions* near that point. The series

$$F(\alpha, \beta; \gamma; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \qquad |z| < 1 \tag{C.5}$$

where $(a)_n := a(a+1)\cdots(a+n-1)$, $(a)_0 := 1$, is the canonical solution of (C.4), corresponding to the characteristic exponent $\rho_1^{(0)} = 0$. The function $F(\alpha, \beta; \gamma; z)$, defined by (C.5) for $|z| < 1$, can be analytically continued for $z \in \mathbb{C}\backslash(1, +\infty)$ [56, 57].

Evidently, $F(\alpha, \beta; \gamma; z) = F(\beta, \alpha; \gamma; z)$. If $\alpha = -m$ or $\beta = -m$, $m = 0, 1, 2, \ldots$, then $F(\alpha, \beta; \gamma; z)$ is a polynomial of degree m.

Another canonical solution of (C.4), corresponding to the characteristic exponent $\rho_2^{(0)} = 1 - \gamma$ for $\gamma \notin \mathbb{N}$, is

$$z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; z).$$

Canonical solutions near the singular point $z = 1$ are

$$F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z)$$

and if $\gamma - \alpha - \beta \notin \mathbb{Z}$ also

$$(1 - z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta; \gamma - \alpha - \beta + 1; 1 - z).$$

There are expansions of $F(\alpha, \beta; \gamma; z)$ through canonical solutions near the singular points $z = 1$ and $z = \infty$ [57, 58], important for spectral problems. The first one, used in the present paper, is

$$\begin{aligned} F(\alpha, \beta; \gamma; z) = {} & \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} F(\alpha, \beta; \alpha + \beta - \gamma + 1; 1 - z) \\ & + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)} (1 - z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, 1 - z), \\ & |\arg(1 - z)| < \pi \end{aligned} \tag{C.6}$$

if $\gamma - \alpha - \beta \notin \mathbb{Z}$. Here, Γ is the *gamma-function*. It has no zeros and has poles of the first order at the points $z = -m$, $m = 0, 1, 2, \ldots$. Its logarithmic derivative $\psi_\Gamma(z) := \Gamma'(z)/\Gamma(z)$ also has poles of the first order at the same points.

For $\gamma - \alpha - \beta \in \mathbb{Z}$, every summand at the right-hand side of (C.6) is singular and it holds for $m = 0, 1, 2, \ldots$

$$\begin{aligned} F(\alpha, \beta; \alpha + \beta + m; z) = {} & \frac{\Gamma(m)\Gamma(\alpha + \beta + m)}{\Gamma(\alpha + m)\Gamma(\beta + m)} \sum_{n=0}^{m-1} \frac{(\alpha)_n(\beta)_n}{n!(1 - m)_n} (1 - z)^n \\ & - \frac{\Gamma(\alpha + \beta + m)}{\Gamma(\alpha)\Gamma(\beta)} (z - 1)^m \sum_{n=0}^{\infty} \frac{(\alpha + m)_n(\beta + m)_n}{n!(n + m)!} (1 - z)^n (\ln(1 - z) \\ & - \psi_\Gamma(n + 1) - \psi_\Gamma(n + m + 1) + \psi_\Gamma(\alpha + n + m) + \psi_\Gamma(\beta + n + m)), \end{aligned} \tag{C.7}$$

$$\begin{aligned} F(\alpha, \beta; \alpha + \beta - m; z) = {} & \frac{\Gamma(m)\Gamma(\alpha + \beta - m)}{\Gamma(\alpha)\Gamma(\beta)} (1 - z)^{-m} \sum_{n=0}^{m-1} \frac{(\alpha - m)_n(\beta - m)_n}{n!(1 - m)_n} (1 - z)^n \\ & - \frac{(-1)^m\Gamma(\alpha + \beta - m)}{\Gamma(\alpha - m)\Gamma(\beta - m)} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{n!(n + m)!} (1 - z)^n \\ & \times (\ln(1 - z) - \psi_\Gamma(n + 1) - \psi_\Gamma(n + m + 1) + \psi_\Gamma(\alpha + n) + \psi_\Gamma(\beta + n)), \\ & |\arg(1 - z)| < \pi, |1 - z| < 1. \end{aligned} \tag{C.8}$$

In the case $\mathrm{Re}\,(\gamma - \alpha - \beta) < 0$, formulae (C.6) – (C.8) imply

$$\lim_{z\to 1} F(\alpha, \beta; \gamma; z)(1 - z)^{\alpha+\beta-\gamma} = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}. \tag{C.9}$$

The Fuchsian equation (C.1) with four singular points by transformations (C.2) and (C.3) can be reduced to the *Heun equation*

$$w''(t) + \left(\frac{\gamma}{t} + \frac{\delta}{t - 1} + \frac{\varepsilon}{t - d}\right) w'(t) + \frac{\alpha\beta t - q}{t(t - 1)(t - d)} w(t) = 0, \tag{C.10}$$

where $0, 1, d, \infty$ are its four singular points ($d \neq 0, 1, \infty$) and $\alpha + \beta - \gamma - \delta - \varepsilon + 1 = 0$. The corresponding P-symbol is

$$P \left\{ \begin{matrix} 0 & 1 & d & \infty & \\ 0 & 0 & 0 & \alpha & ; t \\ 1-\gamma & 1-\delta & 1-\varepsilon & \beta & \end{matrix} \right\}.$$

Note that the *accessory parameter* q does not arise in this P-symbol.

The theory of the Heun equation is much less explicit than the theory of the Riemannian equation. In particular, there are no explicit expressions of canonical solutions near different singular points through each other. Therefore, there are only approximate methods for solving spectral problems connected with the Heun equation, using continued fractions (see for example [59] and references therein).

The substitution $z = P(t)$ for a rational function P transforms equation (C.1) into another Fuchsian equation with generally a greater number of singular points. Therefore, sometimes the inverse transformation can decrease the number of singular points of a Fuchsian equation[4].

At the present time there is no general theory of such reduction. However in [60] there were classified all Heun equations (C.10) that can be obtained by a substitution $z = P(t)$ from the hypergeometric one (C.4). By the inverse transformation, these Heun equations are reduced to hypergeometric equations.

The first condition for existing such reduction is the position of the point d. Let

$$(z_1, z_2, z_3, z_4)_{\text{c.r.}} := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

be the *cross-ratio* of four pairwise distinct points from $\overline{\mathbb{C}}$. It is well known that a cross-ratio is invariant under Möbius transformations. The group $\mathfrak{S}_4$, permuting points z_1, z_2, z_3 and z_4, acts on their cross-ratio. The cross-ratio orbit $\mathcal{O}_{\mathfrak{S}_4}(s)$ of $s := (z_1, z_2, z_3, z_4)_{\text{c.r.}}$ consists of points $s, 1-s, 1/s, 1/(1-s), s/(s-1), (s-1)/s \in \overline{\mathbb{C}}$.

In general position this orbit consists of six points, but there are two exceptional cases: the orbit $-1, \frac{1}{2}, 2$ and the orbit $\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$. If $(z_1, z_2, z_3, z_4)_{\text{c.r.}} \in \left(-1, \frac{1}{2}, 2\right)$, then (z_1, z_2, z_3, z_4) is a *harmonic quadruple*. If $(z_1, z_2, z_3, z_4)_{\text{c.r.}} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathbf{i}$, then (z_1, z_2, z_3, z_4) is an *equianharmonic quadruple*.

Points of a harmonic quadruple lie on a circle or on a line. By a Möbius transformation they can be mapped into vertices of a square in $\mathbb{C}$. If (z_1, z_2, z_3, ∞) is a harmonic quadruple, then (z_1, z_2, z_3) are collinear, equally spaced points. If (z_1, z_2, z_3, ∞) is an equianharmonic quadruple, then (z_1, z_2, z_3) are vertices of an equilateral triangle in $\mathbb{C}$.

Theorem C.1 ([60]). *All cases, when nontrivial Heun equation (C.10) (i.e., $\alpha\beta \neq 0$ or $q \neq 0$) can be obtained from the hypergeometric one (C.4) by the rational substitution $z = P(t)$, are as follows:*

1. *Harmonic case: $d \in \mathcal{O}_{\mathfrak{S}_4}(2)$. Suppose $d = 2$,[5] then $q/(\alpha\beta)$ must be equal 1, and characteristic exponents of points $t = 0$ and $t = d = 2$ must be the same, i.e. $\gamma = \varepsilon$. The function $P(t)$ is a degree-2 polynomial and can be chosen as $P(t) = t(2-t) = 1-(t-1)^2$. It maps $t = 0, 2$ to $z = 0$ and $t = 1$ to $z = 1$.[6]*
   If additionally $1 - \delta = 2(1-\gamma)$, then $P(t)$ can be chosen also as degree-4 polynomial $4\left(t(2-t) - \frac{1}{2}\right)^2$, which maps $t = 0, 1, 2$ to $z = 1$.

[4] Generally, the inverse transformation does not conserve the Fuchs class of differential equations.
[5] If $d \in \mathcal{O}_{\mathfrak{S}_4}(s)$, then the quadruple $(0, 1, d, \infty)$ can be mapped into the quadruple $(0, 1, s, \infty)$ by a Möbius transformation, which also transforms parameters of equation (C.10).
[6] This transformation was found already in [61].

2. $d \in \mathcal{O}_{\mathfrak{S}_4}(4)$. *Suppose* $d = 4$, *then* $q/(\alpha\beta)$ *must be equal 1, characteristic exponents of the point* $t = 1$ *must be double those of the point* $t = d = 4$, *i.e.* $1 - \delta = 2(1 - \varepsilon)$, *and* $t = 0$ *must have characteristic exponents* $0, 1/2$, *i.e.* $\gamma = \frac{1}{2}$. *The function* $P(t)$ *is a degree-3 polynomial and can be chosen as* $(t-1)^2\left(1 - \frac{t}{4}\right)$. *It maps* $t = 0$ *to* $z = 1$ *and* $t = 1, 4$ *to* $z = 0$.
3. *Equianharmonic case:* $d \in \mathcal{O}_{\mathfrak{S}_4}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}\right)$. *Characteristic exponents of points* $t = 0, 1, d$ *are the same, i.e.* $\gamma = \delta = \varepsilon$. *Suppose* $d = \frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$, *then* $q/(\alpha\beta)$ *must be equal* $\frac{1}{2} + \frac{\sqrt{3}}{6}\mathbf{i}$. *The function* $P(t)$ *is a degree-3 polynomial and can be chosen as* $\left(1 - t\Big/\left(\frac{1}{2} + \frac{\sqrt{3}}{6}\mathbf{i}\right)\right)^3$. *It maps* $t = 0, 1, d$ *to* $z = 1$ *and* $t = q/(\alpha\beta)$ *to* $z = 0$, *thus creating a new singular point. If additionally* $\gamma = \delta = \varepsilon = \frac{2}{3}$, *then* $P(t)$ *can be chosen also as degree-6 polynomial*
$$4\left(\left(1 - \frac{t}{\frac{1}{2} + \frac{\sqrt{3}}{6}\mathbf{i}}\right)^3 - \frac{1}{2}\right)^2,$$
which maps $t = 0, 1, d, q/(\alpha\beta)$ *to* $z = 1$.
4. $d \in \mathcal{O}_{\mathfrak{S}_4}\left(\frac{1}{2} + \frac{5\sqrt{2}}{4}\mathbf{i}\right)$. *Suppose* $d = \frac{1}{2} + \frac{5\sqrt{2}}{4}\mathbf{i}$, *then* $q/(\alpha\beta)$ *must be equal* $\frac{1}{2} + \frac{\sqrt{2}}{4}\mathbf{i}$, *characteristic exponents of the point* $t = d$ *must be* $0, 1/3$, *i.e.* $\varepsilon = 2/3$, *and points* $t = 0, 1$ *must have characteristic exponents* $0, 1/2$, *i.e.* $\gamma = \delta = 1/2$. *The function* $P(t)$ *is a degree-4 polynomial and can be chosen as*
$$\left(1 - \frac{t}{\frac{1}{2} + \frac{5\sqrt{2}}{4}\mathbf{i}}\right)\left(1 - \frac{t}{\frac{1}{2} + \frac{\sqrt{2}}{4}\mathbf{i}}\right)^3.$$
It maps $t = 0, 1$ *to* $z = 1$ *and* $t = d, q/(\alpha\beta)$ *to* $z = 0$.
5. $d \in \mathcal{O}_{\mathfrak{S}_4}\left(\frac{1}{2} + \frac{11\sqrt{15}}{90}\mathbf{i}\right)$. *Suppose* $d = \frac{1}{2} + \frac{11\sqrt{15}}{90}\mathbf{i}$, *then* $q/(\alpha\beta)$ *must be equal* $\frac{1}{2} + \frac{\sqrt{15}}{18}\mathbf{i}$, *characteristic exponents of the point* $t = d$ *must be* $0, 1/2$, *i.e.* $\varepsilon = 1/2$, *and points* $t = 0, 1$ *must have characteristic exponents* $0, 1/3$, *i.e.* $\gamma = \delta = 2/3$. *The function* $P(t)$ *is a degree-5 polynomial and can be chosen as*
$$-\mathbf{i}\frac{2025\sqrt{15}}{64}t(t-1)\left(t - \frac{1}{2} - \frac{\sqrt{15}}{18}\mathbf{i}\right)^3.$$
It maps $t = 0, 1, q/(\alpha\beta)$ *to* $z = 0$ *and* $t = d$ *to* $z = 1$.

Note that there are three independent parameters in the first case of theorem (C.1) (for example, α, β, γ) and all other cases contain only one or two free parameters. It means that the first case is more rife in applications. In fact, it is the only one which occurs in the present paper.

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